The Calm Policymaker:

Imperfect Common Knowledge in New Keynesian Models

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Abstract

Determinacy is ensured in the New Keynesian model when firms face imperfect common knowledge, regardless of whether the Taylor principle is satisfied. Strategic complementarity in pricing and idiosyncratic noise in firms' signals, however small, are together sufficient to eliminate backward-looking solutions and rational bubbles without appealing to the assumptions of Blanchard and Kahn (1980). Standard solutions emerge when the Taylor principle is followed, but when it is not, the price level – and not just inflation – is stationary. Indeed, a unique and stable solution emerges when the interest rate is pegged to its steady-state value (contra Sargent and Wallace, 1975) or when the trend rate of inflation is increased (contra Ascari and Sbordone, 2014).

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1 Introduction

A long literature has studied the question of price level determinacy, dating to the rise of the rational expectations paradigm,[1] with Sargent and Wallace's (1975) demonstration of indeterminacy in a model with rational expectations under an interest rate peg. It is now widely accepted that when monetary policy is set via interest rates, determinacy and stability rely critically on the Taylor principle: that when inflation rises, the nominal interest rate should be raised sufficiently – usually by more than one-for-one – to ensure that the real interest rate will rise, thus damping demand and lowering inflation. Formally, when a New Keynesian (NK) model is closed with an interest rate rule and solved with the assumptions introduced by Blanchard and Kahn (1980), a lower bound emerges on the central bank's marginal response to inflation for the solution to be unique.[2] Absent the Taylor principle, the textbook model admits the possibility of sunspot shocks to select between solutions, thereby adding to the volatility of the economy.

By contrast, this paper argues that it is not strictly necessary for a central bank to respond to temporary deviations of the economy from its long run trend: that the policymaker's response can be calm, instead of forceful, without threatening the stability of the economy. This is not to suggest that policy ought not respond, or that if it does it will be ineffective. The model below adopts the basic NK framework, with policy operating through the same channels, and with equal effect. Nevertheless, this paper's results partially confound such discussion by demonstrating the determinacy of (deviations from trend in) the price level when arbitrarily small amounts of noise are introduced into firms' information sets, regardless of the strength of the central bank's response to inflation.[3]

The subdued responsiveness of interest rates in the post-crisis era, combined with the relative stability of inflation over the same period, poses a puzzle for the NK model. Although it is possible that central banks' asset purchases substituted for changes in short-term interest rates, the efficacy of such 'quantitative easing' (QE) is far from certain.[4] Furthermore, even accepting that unconventional policy can successfully substitute for conventional policy, it seems plausible that the combined effect of the two did not satisfy the Taylor principle in the post-crisis period (table 1 reports estimated Taylor rules in the pre- and post-crisis periods, using a shadow rate to account for the effect of unconventional policy in the latter).

[1] The broader question of what *determines* an economy's price level is clearly far older, dating (at least) to Hume's (1748) advocacy of the quantity theory of money.

[2] The Blanchard-Kahn conditions are not without challenge. For example, Cochrane (2011) critiques their use in solving the NK model, arguing *inter alia* that the Taylor principle cannot be *ex ante* credible, as it is designed to produce explosive inflation in the event of off-equilibrium behaviour and policymakers would retain *ex post* options for bringing inflation back in check.

[3] The model I present is linearised around a deterministic trend, implying an assumption that long-run inflation expectations remain anchored throughout.

[4] See, for example, Eggertsson and Woodford (2003), Cúrdia and Woodford (2011) or Reis (2016).

Period	OLS			GMM			Std. dev. of inflation
	ϕ_π	ϕ_y	ρ	ϕ_π	ϕ_y	ρ	
1979:3 – 2008:2	2.09	1.45	0.88	2.52	0.83	0.83	2.07
	(0.45)	(0.67)	(0.04)	(0.63)	(0.38)	(0.04)	
2008:3 – 2015:4	-0.07	0.20	0.83	-2.39	0.53	0.87	0.86
	(0.55)	(0.40)	(0.08)	(2.96)	(1.10)	(0.14)	

Note: The estimated rule is $r_t = \rho r_{t-1} + (1-\rho)(\bar{r} + \phi_\pi \pi_t + \phi_y y_t) + \varepsilon_t$, where r_t is the effective federal funds rate, or the shadow rate of Wu and Xia (2016) in the post-crisis period; π_t is the annualised quarterly change in the GDP deflator; and y_t is the CBO-implied output gap. Robust standard errors are reported in parentheses. Following Clarida, Galí and Gertler (2000), for the GMM estimation lags of the interest rate, inflation, output gap, commodity price inflation, money growth and the spread between 10-year and 3-month US treasuries are used as instruments. See appendix A for detail.

Table 1: Estimated Taylor rules

Inspired by the near-constant interest rates and persistently below-target inflation rates seen in the post-crisis era, a number of authors have proposed a 'neo-Fisherian' view of inflation. In this framework, when the interest rate is pegged below its original steady state value forever, inflation does not explode but, instead, falls to accommodate the change.[5] In one formalisation, Cochrane (2017) has highlighted a 'backward stable' solution to the NK model when the Blanchard-Kahn conditions are not met. A number of authors have responded to Cochrane's proposal but, prior to this paper, none have been able to reject his chosen solution while retaining rational expectations.[6]

Extending the textbook three-equation NK model[7] to impose *Imperfect Common Knowledge* (ICK) on firms – each rationally combining idiosyncratically noisy signals of the underlying state of the economy while facing strategic complementarity in their price-setting – I establish the following results:

1. *Uniqueness.* So long as firms never discover past values of the price level with certainty, backward-looking solutions and extrinsic bubbles are eliminated without appealing to the conditions of Blanchard and Kahn (1980).

2. *Standard results remain.* The solution is a pertubation from the purely-forward solution under full information. It nests the canonical solution when the Taylor principle is satisfied and firms' noise is taken towards zero.

3. *Interest rate peg.* In partial contrast to the results of Sargent and Wallace (1975), a unique and stable solution exists when the nominal interest rate remains pegged

[5]The proposal rests on the Fisher relation, which emerges from an Euler equation when at trend. It is therefore closely related to the 'liquidity trap' of Benhabib, Schmitt-Grohe and Uribe (2001).

[6]In particular, García-Schmidt and Woodford (2015) propose a model with iterative, but incomplete, revisions of beliefs each period; Gabaix (2016) describes a model in which agents pay reduced attention to specific variables when forecasting; and Evans and McGough (2018) examine the learnability of Cochrane's 'backward stable' solution versus the 'minimum state variable' solution.

[7]See, for example, Woodford (2003*a*) or Galí (2008).

at its steady-state level.[8]

4. *Stationary prices.* When the central bank declines to satisfy the Taylor principle, the price level – and not just inflation – is stationary around its trend path, with policymaker-determined persistence. This ensures that the real interest rate always rises following a positive demand shock, regardless of the central bank's response.[9]

5. *Trend inflation.* A unique solution still emerges in the presence of positive trend inflation, regardless of the level of that trend or the coefficients of the central bank's decision rule.

6. *Inflation and output volatility.* The unconditional volatility of inflation peaks at the Taylor threshold, falling as the central bank's marginal response to inflation moves in either direction. In contrast, output volatility is generally larger in a 'passive' monetary regime and falls only when in an 'active' regime.[10]

7. *Avoiding a liquidity trap.* When the central bank declines to satisfy the Taylor principle, there exists a single, globally stable, steady-state equilibrium. The liquidity trap emphasised by Benhabib, Schmitt-Grohe and Uribe (2001, 2002) is therefore avoided, even when a lower bound on interest rates is respected.

In short, without common knowledge, firms are unable to co-ordinate on past states of the economy. Since backward-looking solutions are functions of past states by definition, they become ineligible in a setting with ICK (similar arguments apply to rational bubbles). The purely forward-looking solution, if one exists, is therefore automatically unique regardless of the parameters that underlie the eigenvalues of the system.

One possible interpretation of this result is that ICK represents an equilibrium selection device in an otherwise full-information model by positing an epsilon of idiosyncratic noise in firms' information sets – sufficiently small as to not affect the dynamics of the model, but still positive so as to ensure uniqueness without appealing to the Blanchard-Kahn conditions. In addition to rejecting the 'backward stable' solution of Cochrane (2017), this paper therefore also poses challenges to empirical studies that rely on the possibility of sunspot shocks in a model with rational expectations, such as Lubik and Schorfheide (2004) or Ascari, Bonomolo and Lopes (2016).

Solution uniqueness in (static) models when agents have heterogeneous information and a continuous choice set is well established,[11] but its implications for equilibrium selection in DSGE models do not appear to have been widely appreciated. Hellwig

[8]When pegged away from its original steady state, whether the nominal economy explodes or conforms depends on whether agents change their view of trend inflation.

[9]The real interest falls on impact under an interest rate peg, but subsequently rises above, and remains above, trend thereafter, with the integral over time being positive.

[10]I use the terminology of Leeper (1991).

[11]See, for example, Morris and Shin (1998).

and Veldkamp (2009), for example, emphasise that the 'sticky information' models of Mankiw and Reis (2002) and Reis (2006)[12] feature unique solutions, but do not explore the implications for parameter regions that do not satisfy the Blanchard-Kahn conditions.

Methodologically, this paper adds to the ICK literature by deriving an exact finite-state representation that accommodates both dynamic elements in agents' decision rules and endogenous signals. By contrast, earlier work has either (i) approximated the solution by granting agents full knowledge of the state with a T-period lag (e.g. Lorenzoni, 2009) or by truncating the hierarchy of beliefs (e.g. Nimark, 2017); or (ii) produced a finite-state representation only when agents face a sequence of static problems with exogenous signals (e.g. Woodford, 2003*b*). More recently, Huo and Takayama (2016) have demonstrated a finite-state representation in models with dynamic choices when agents' signals are exogenous and proven the impossibility of a finite representation when agents observe contemporaneous endogenous signals. The method used here is arguably simpler than that of Huo and Takayama (2016) and successfully includes endogenous signals by having them be observed with a lag.

Related literature. This is by no means the first paper to apply ICK to the study of monetary business cycles,[13] but to my knowledge it is the first to examine the implications for determinacy. Woodford (2003*b*) first introduced Townsend's (1983) hierarchy of expectations to a nominal economy, using a reduced-form expression for demand and demonstrating sluggish aggregate behaviour following a shock to nominal spending, despite price flexibility. Nimark (2008) extends Woodford's approach to include a standard demand side to the economy, but grants firms perfect knowledge of the previous period's price level. This maintains the possibility of indeterminacy and so requires approaches like the Taylor principle to address it. Melosi (2014) estimates a similar model for the US economy. More recently, Kohlhas (2014) has re-explored the 'anti-disclosure' result of Morris and Shin (2005), while Angeletos and Lian (2016*b*) have demonstrated that the absence of perfect common knowledge can address the forward guidance 'puzzle' of Del Negro, Giannoni and Patterson (2016). Adam (2007) and Angeletos and La'O (2017) study optimal policy in the presence of ICK among firms.

It may also help to be clear on what the model of this paper *is not*. The content and precision of firms' signals are set here as a modelling choice, rather than chosen endogenously by the firms themselves. I therefore avoid the complexity of strategic information acquisition, as in Sims (2003), Mackowiak and Wiederholt (2009) or Hellwig, Kohls and Veldkamp (2012). All of the firms' signals are assumed to include some degree of idiosyncratic noise, so there are no purely public signals of the sort examined by Morris

[12] Wherein agents occasionally discover the true state of the economy, but otherwise acquire no new information from period to period.

[13] Angeletos and Lian (2016*a*) provide a recent overview of models of incomplete information, including imperfect common knowledge.

and Shin (2002) and the literature that followed. The strategy space of firms – their (relative) price – is also continuous rather than countable, thus avoiding the correlated equilibria of Aumann (1987) or the 'private sunspot' concept of Angeletos (2008). Finally, the indeterminacy obtained in the models of Lubik and Matthes (2016) and Mertens, Matthes and Lubik (2017) is avoided here because multiple agents (individual firms) have non-nestable information sets, thus giving rise to a hierarchy of expectations.

Paper structure. The rest of the paper is arranged as follows. Section 2 first illustrates the achievement of determinacy with ICK in a toy model. Section 3 then presents the basic NK model under ICK, before section 4 presents the solution under both full information and ICK. Section 5 presents a variety of implications that follow, conditional on the model, and section 6 extends the basic model to add positive trend inflation. Finally, section 7 discusses the implications for steady-state equilibria and the possibility of liquidity traps before section 8 concludes.

2 A toy model

To help illustrate the core concepts of the paper, before turning to the New Keynesian model I first examine determinacy in the smallest possible dynamic model with rational expectations and imperfect common knowledge. In it, a continuum of agents, indexed $j \in [0,1]$, have individual decision rules given by:

$$z_t(j) = E_t(j)[x_t] + \beta E_t(j)[z_{t+1}] \tag{1}$$

where $z_t \equiv \int_0^1 z_t(j)\,dj$ is their average action; $E_t(j)[\cdot] \equiv E[\cdot|\mathcal{I}_t(j)]$ is the Bayes-rational expectation of agent j, subject to their information set in period t; and x_t is an unobserved, exogenous driving process:

$$x_t = \rho x_{t-1} + u_t \quad \text{where} \quad \rho \in (0,1) \quad \text{and} \quad u_t \sim N\left(0, \sigma_u^2\right) \tag{2}$$

Taking the average of (1) gives the equilibrium condition of the model:

$$z_t = \overline{E}_t[x_t] + \beta \overline{E}_t[z_{t+1}] \tag{3}$$

where $\overline{E}_t[\cdot] \equiv \int_0^1 E_t(j)[\cdot]\,dj$ is the average expectation of individual agents. Each period, agents observe signals of x_t that are unbiased but idiosyncratically noisy, so that their information sets evolve as:

$$\mathcal{I}_t(j) = \{\mathcal{I}_{t-1}(j), s_t(j)\} \tag{4}$$

$$s_t(j) = x_t + v_t(j) \quad \text{where} \quad v_t(j) \overset{\text{i.i.d.}}{\sim} N\left(0, \sigma_v^2\right) \tag{5}$$

Agents' noise shocks, $v_t(j)$, are perfectly transitory and independent of both each other and the underlying state (x_t).

2.1 Solving the toy model under full information

Agents have full information when $\sigma_v = 0$. In that case, equation (3) simplifies to:

$$z_t = x_t + \beta E_t^{\Omega}\left[z_{t+1}\right] \tag{6}$$

where $E_t^{\Omega}[\cdot] \equiv E[\cdot|\Omega_t]$ and Ω_t is the set of all information in existence in period t. The model of (2) and (6) is widely known to feature an infinite number of solutions. In particular, following the notation of Blanchard (1979), there exists: (i) a *purely forward-looking solution* obtained by substituting (6) forward:

$$z_t^{(F)} = \left(\sum_{s=0}^{\infty}(\beta\rho)^s\right) x_t \tag{7a}$$

(ii) a *purely backward-looking solution* obtained by supposing perfect foresight in (6):

$$z_t^{(B)} = \frac{1}{\beta}\left(z_{t-1} - x_{t-1}\right); \text{ and} \tag{7b}$$

(iii) a *rational bubble* as any process that satisfies (6) with the fundamental removed:

$$\theta_t = \beta E_t^{\Omega}\left[\theta_{t+1}\right] \tag{7c}$$

All told, the full set of solutions is given by:

$$z_t = \xi_t z_t^{(B)} + (1-\xi_t)\, z_t^{(F)} + \theta_t \quad \text{where} \quad \xi_t \in \mathbb{R} \tag{7d}$$

The elements ξ_t and θ_t are referred to as *sunspot shocks* – processes extrinsic to the model which nevertheless feature in the solution.[14] Without further assumptions to pin down the paths for ξ_t and θ_t, the model is therefore indeterminate. When a unique solution is required,[15] the usual approach is to adopt the conditions of Blanchard and Kahn (1980):

A1: The solution must be stationary.

A2: $\left|\frac{1}{\beta}\right| > 1$

where assumption A2 is a parameter restriction designed to ensure that the purely backward-looking solution and the rational bubble are both explosive.[16] It is easy to see that the set of solutions (7d) can then only satisfy both A1 and A2 when $\xi_t = \theta_t = 0\,\forall t$, so that the solution simplifies to:

$$z_t = \left(\sum_{s=0}^{\infty}(\beta\rho)^s\right) x_t = \left(\frac{1}{1-\beta\rho}\right) x_t \tag{8}$$

where the second equality requires that $|\beta\rho| < 1$. If this is not satisfied, the purely forward-looking solution does not exist.

[14] Although appreciation of indeterminacy came earlier, the idea of interpretting ξ_t and θ_t as stochastic processes that influence the real economy dates to Cass and Shell (1983).

[15] For a recent example of a model that embraces the multiplicity of solutions, see Ascari, Bonomolo and Lopes (2016), who use sunspot shocks to explain the volatility of inflation in the pre-Volcker era.

[16] If z_t was a vector and β a matrix, the condition would be that β^{-1} have eigenvalues outside the unit circle for each forward-looking variable in z_t.

2.2 Solving the toy model under imperfect common knowledge

When agents' signals contain idiosyncratic noise, the law of iterated expectations breaks down ($\overline{E}_t\left[\overline{E}_t\left[x_t\right]\right] \neq \overline{E}_t\left[x_t\right]$) and it becomes necessary to consider the hierarchy of their average expectations. We may compactly define this hierarchy recursively as:

$$X_t \equiv \begin{bmatrix} x_t \\ \overline{E}_t\left[X_t\right] \end{bmatrix} \tag{9}$$

which is an infinite-dimension vector.[17] I refer to X_t as the *full state* of the model to distinguish it from the underlying state, x_t. It is helpful, too, to define S and T as the selection matrices such that $SX_t = x_t$ and $TX_t = \overline{E}_t\left[X_t\right]$. A solution is then a reduced-form expression for z_t as a function of X_t, potentially with lags, and a law of motion for X_t that describes agents' average belief formation. As before, I first consider the purely forward-looking solution and then consider the possibility of backward-looking solutions.

Proposition 1. *For the simple model under universally dispersed information,*

a) the purely forward-looking solution exists (is finite) when $\beta\rho < 1$ and has the form

$$z_t = \boldsymbol{\gamma}' X_t \quad \textit{where} \quad \boldsymbol{\gamma}' = ST\left(I - \beta FT\right)^{-1} \tag{10a}$$

$$X_t = FX_{t-1} + Gu_t \tag{10b}$$

b) the forward solution under idiosyncratic noise (10) nests the forward solution under full information (7a) when $\sigma_v^2 = 0$ and approaches it smoothly as $\sigma_v^2 \to 0$.

Proof. See appendix B. ■

Since x_t is autoregressive of order 1, individual agents' Bayes-optimal estimator regarding it is a Kalman filter. Given the recursive nature of the Kalman filter, x_t and $\overline{E}_t\left[x_t\right]$ therefore jointly follow a vector AR(1) process. A Kalman filter is thus also optimal for estimating $\overline{E}_t\left[x_t\right]$, and so on up the hierarchy, giving the vector AR(1) process for X_t in the solution. In the appendix I show that the matrices F and G are given by:

$$F = \rho \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ k_1 & (1-k_1) & 0 & 0 & \cdots \\ k_2 & (k_1-k_2) & (1-k_1) & 0 & \cdots \\ k_3 & (k_2-k_3) & (k_1-k_2) & (1-k_1) & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ k_1 \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} \tag{11}$$

where k_q is the Kalman gain used to form agents' q^{th}-order expectation regarding x_t.

[17] An equivalent, but more notationally complex definition of the full state is $X_t \equiv \begin{bmatrix} x_{t|t}^{(0)} & x_{t|t}^{(1)} & x_{t|t}^{(2)} & \cdots \end{bmatrix}'$, where the 0th-order expectation of a variable is the variable itself; the 1st-order expectation is agents' average expectation of the variable; the 2nd-order expectation is agents' average expectation about the 1st-order expectation; and so on: $x_{t|t}^{(0)} \equiv x_t$ and $x_{t|t}^{(q)} \equiv \overline{E}_t\left[x_{t|t}^{(q-1)}\right]$ $\forall q \geq 1$.

2.2.1 Rejecting rational bubbles

With the purely forward-looking solution established, I now turn to consider various possible sunspots. One possibility, excluded in the above because of the the tightly specified signal vector, is that of a rational bubble:

$$z_t = \boldsymbol{\gamma}' X_t + \theta_t \tag{12a}$$

$$\theta_t = \beta E_t^{\Omega} [\theta_{t+1}] \tag{12b}$$

where θ_t is independent of u_s for all s, t. This solution clearly cannot be rejected if θ_t is perfectly observed by all agents, even if signals of x_t are universally observed with noise:

$$\boldsymbol{s}_t(i) = \begin{bmatrix} x_t + v_t(i) \\ \theta_t \end{bmatrix} \ \forall i \tag{13}$$

In that case, backward-looking solutions would be eliminated, as described in section 2.2.2 below, but the rational bubble would remain. Instead, I consider a situation where some fraction of agents, $\phi \in (0, 1)$, observe the candidate bubble with idiosyncratic noise:

$$\boldsymbol{s}_t(i) = \begin{cases} \begin{bmatrix} x_t + v_t(i) \\ \theta_t + e_t(i) \end{bmatrix} & \text{where} \quad e_t(i) \overset{\text{i.i.d.}}{\sim} N(0, \sigma_e^2) \qquad \text{if } i \in [0, \phi) \\ \begin{bmatrix} x_t + v_t(i) \\ \theta_t \end{bmatrix} & \qquad\qquad\qquad\qquad\qquad\qquad \text{if } i \in [\phi, 1] \end{cases} \tag{14}$$

The idiosyncratic noise $e_t(i)$ is assumed to be orthogonal to x_s, θ_s and $v_s(j)$ $\forall i, j, s, t$. In this case, with common knowledge removed, it follows that the bubble term is eliminated:

Proposition 2. *For the toy model under imperfect common knowledge with agents' signals given by (14), rational bubbles are eliminated if $\phi > 0$ and $\sigma_e > 0$.*

Proof. Substituting the candidate solution (12a) into the equilibrium condition (3) yields:

$$z_t = \boldsymbol{\gamma}' X_t + \beta \left(\phi \overline{E}_t^{\theta} [\theta_{t+1}] + (1 - \phi) E_t^{\Omega} [\theta_{t+1}] \right) \tag{15}$$

where $\overline{E}_t^{\theta} [\cdot]$ is the average expectation of agents that are imperfectly informed about θ. Making use of (12b) and the fact that $\overline{E}_t^{\theta} [\cdot] = \overline{E}_t^{\theta} \left[E_t^{\Omega} [\cdot] \right]$ then gives:

$$z_t = \boldsymbol{\gamma}' X_t + \theta_t + \phi \left\{ \overline{E}_t^{\theta} [\theta_t] - \theta_t \right\} \tag{16}$$

But, by comparison to (12a), this then requires that either $\phi = 0$ or that $\sigma_e = 0$ (so that $\overline{E}_t^{\theta} [\theta_t] = \theta_t$), either of which would imply a contradiction. More generally, consider candidate solutions of the entire hierarchy of agents' expectations about the bubble:

$$z_t = \boldsymbol{\gamma}' X_t + \boldsymbol{a}' \Theta_t \quad \text{where} \quad \Theta_t \equiv \begin{bmatrix} \theta_t \\ \overline{E}_t [\Theta_t] \end{bmatrix} \tag{17}$$

In a linear model, Θ_t will follow an AR(1) process:

$$\Theta_t = Q\Theta_{t-1} + R\varepsilon_t \tag{18}$$

where ε_t is the innovation driving θ_t, while Q and R are matrices of parameters based on agents' filtering with equivalent structures to F and G (B.7b). In particular, Q will be lower triangular and have eigenvalues less than or equal to $1/\beta$ (see the derivation of F in the appendix). Substituting this into the equilibrium condition (3) then produces a requirement that:

$$\boldsymbol{a}'\left(I - \beta QT\right) = \mathbf{0} \tag{19}$$

If all agents had full information, this would be equivalent to $\boldsymbol{a}'\left(I - \beta Q\right) = \mathbf{0}$ since all the elements of Θ_t would be identical. With Q having an eigenvalue of $1/\beta$, the term $(I - \beta Q)$ would be singular and there would be an infinite number of possible values for $\boldsymbol{a}$. With some agents not observing the bubble perfectly, however, $(I - \beta QT)$ is invertible (its rows are linearly independent), implying that $\boldsymbol{a} = \mathbf{0}$. ■

This demonstrates that the existence of a bubble is a knife-edged result. It is rejected whenever a positive measure of agents, no matter how small, have idiosyncratic noise in their signals of the bubble, no matter how small. Importantly, the correlated equilibrium concept of Aumann (1987) (for a more recent application of a similar concept, see Angeletos, 2008) does not apply in this context because the strategy space for $z_t(i)$ is continuous on the real line rather than discrete, and so uncountable.

2.2.2 Rejecting backward-looking solutions

I next consider the possibility of backward-looking solutions – that is, alternatives to (10a) that feature additional weight on past fundamentals. To begin, it is helpful to first consider as a candidate the ICK equivalent of the purely-backward looking solution under full information:

$$z_t = \frac{1}{\beta}\left(z_{t-1} - \overline{E}_{t-1}\left[x_{t-1}\right]\right) \tag{20}$$

where x_{t-1} has been replaced with $\overline{E}_{t-1}\left[x_{t-1}\right]$. Stepping this candidate solution forward by one period and taking the average period-t expectation gives:

$$\overline{E}_t\left[z_{t+1}\right] = \frac{1}{\beta}\overline{E}_t\left[z_t - \overline{E}_t\left[x_t\right]\right] \tag{21}$$

When agents share a common information set, as they do under full information,[18] they know the average action perfectly ($\overline{E}_t\left[z_t\right] = z_t$) and the law of iterated expectations

[18]Strictly, this only requires that agents' information sets be common – they could still be incomplete with a single, shared signal about x_t. See, for example, by Currie, Levine and Pearlman (1986) or Levine et al. (2012).

applies ($\overline{E}_t\left[\overline{E}_t\left[x_t\right]\right] = \overline{E}_t\left[x_t\right]$) so that this would then recover the equilibrium condition of the model. With imperfect common knowledge, however, neither of these apply, so that (20) is rejected as a specific candidate solution.

More generally, though, it is necessary to consider all possible solutions as linear functions of any current or past values of u_t:

$$z_t = \boldsymbol{\alpha}\left(L\right)u_t = \textstyle\sum_{s=0}^{\infty}\boldsymbol{\alpha}_s u_{t-s} \tag{22}$$

This necessarily includes past values of z_t and any average expectation, of any order and formed at any time, about any variable in the past. For the proof, however, I consider solutions as functions of (lags of) X_t:

$$z_t = \boldsymbol{\delta}\left(L\right)X_t = \textstyle\sum_{s=0}^{\infty}\boldsymbol{\delta}_s X_{t-s} \tag{23}$$

This is admisible because (i) $u_t = x_t - \rho x_{t-1}$; (ii) x_t is nested within X_t; and (iii) we are not here envisioning any change in agents' signals, so the law of motion for X_t (10b) will be invariant to the solution (23). As such, for any given set of $\boldsymbol{\delta}$ coefficients, the corresponding $\boldsymbol{\alpha}$ coefficients in (22) can be recovered and *vice versa*.

Proposition 3. *For the toy model, backward-looking solutions are rejected — that is, candidate solutions of the form of (23) all imply that $\boldsymbol{\delta}_q = \mathbf{0}\ \forall q \geq 1$ and $\boldsymbol{\delta}_0 = \boldsymbol{\gamma}$ — without recourse to the Blanchard-Kahn conditions.*

Proof. Stepping the candidate solution forward, taking the period-t expectation and substituting it into the equilibrium condition (3) gives:

$$z_t = \left(S + \beta\left(\boldsymbol{\delta}_0' F + \boldsymbol{\delta}_1'\right)\right)\overline{E}_t\left[X_t\right] + \beta\boldsymbol{\delta}_2'\,\overline{E}_t\left[X_{t-1}\right] + \beta\boldsymbol{\delta}_3'\,\overline{E}_t\left[X_{t-2}\right] + \cdots \tag{24}$$

Under full information, there would be no expectation operators around the X_t, X_{t-1}, etc., which would imply that $\boldsymbol{\delta}_q = \frac{1}{\beta}\boldsymbol{\delta}_{q-1}\ \forall q \geq 1$, with indeterminacy for $\boldsymbol{\delta}_0$. With incomplete information, however, we must evaluate the expectations as functions of current and past states. Given its definition, the period-t expectation about X_t is nested within X_t itself: $\overline{E}_t\left[X_t\right] = TX_t$. For expectations about earlier state vectors, the optimal average Kalman smoother must update each period as:

$$\overline{E}_t\left[X_{t-q}\right] = \overline{E}_{t-1}\left[X_{t-q}\right] + \boldsymbol{m}_q\left\{SX_t - SF\overline{E}_{t-1}\left[X_{t-1}\right]\right\} \tag{25}$$

where $\boldsymbol{m}_q$ is the gain applied to period-t signals when updating beliefs about X_{t-q}. Note

that $\boldsymbol{m}_q \to 0$ as $\sigma_v \to 0$. Substituting (25) for each q into (24) and gathering terms:

$$z_t = \begin{pmatrix} ST + \boldsymbol{\delta}_0' \{\beta FT\} \\ + \boldsymbol{\delta}_1' \{\beta T\} \\ + \boldsymbol{\delta}_2' \{\beta \boldsymbol{m}_1 S\} \\ + \cdots \end{pmatrix} X_t + \begin{pmatrix} \boldsymbol{\delta}_2' \Phi_1 \\ + \boldsymbol{\delta}_3' \Phi_2 \\ + \boldsymbol{\delta}_4' \Phi_3 \\ + \cdots \end{pmatrix} X_{t-1} + \begin{pmatrix} \boldsymbol{\delta}_3' \Phi_1 \\ + \boldsymbol{\delta}_4' \Phi_2 \\ + \boldsymbol{\delta}_5' \Phi_3 \\ + \cdots \end{pmatrix} X_{t-2} + \cdots \tag{26a}$$

$$\text{where } \Phi_1 \equiv \beta \left(\quad T - \boldsymbol{m}_1 SFT \right) \tag{26b}$$

$$\Phi_2 \equiv \beta \left(\boldsymbol{m}_1 S - \boldsymbol{m}_2 SFT \right) \tag{26c}$$

$$\Phi_3 \equiv \beta \left(\boldsymbol{m}_2 S - \boldsymbol{m}_3 SFT \right) \tag{26d}$$

$$\vdots$$

Comparing (26a) to (23), it then follows that:

$$\boldsymbol{\delta}_q' = \sum_{i=1}^{\infty} \boldsymbol{\delta}_{q+i}' \Phi_i \quad \forall q \geq 1 \tag{27}$$

Given the symmetry of this requirement, it must be the case that $\boldsymbol{\delta_1} = \boldsymbol{\delta_2} = \boldsymbol{\delta_3} = \cdots$ (to see this, ignore $q = 1$ and relabel $\boldsymbol{\gamma}_1 = \boldsymbol{\delta}_2, \boldsymbol{\gamma}_2 = \boldsymbol{\delta}_3$, etc; the recursive conditions for $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \cdots$ are identical to those for $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \cdots$, so the solutions must be the same). Imposing this then gives:

$$\boldsymbol{\delta}_q' \left(I - \sum_{i=1}^{\infty} \Phi_i \right) = \mathbf{0} \quad \forall q \geq 1 \tag{28}$$

Evaluating the sum and after expanding F (B.7b), I arrive at:

$$\sum_{i=1}^{\infty} \Phi_i = \beta \left(T + \begin{bmatrix} \mathbf{0}_{\infty \times 1} & \rho \left(\sum_{i=1}^{\infty} \boldsymbol{m}_i \right) & \mathbf{0}_{\infty \times \infty} \end{bmatrix} \right) \tag{29}$$

It is then straightforward to confirm that $(I - \sum_{i=1}^{\infty} \Phi_i)$ is non-singular (its rows are linearly independent). Hence, by (28), it follows that $\boldsymbol{\delta}_q = \mathbf{0}$ $\forall q \geq 1$. Substituting this into (26a) then gives $\boldsymbol{\delta}_0' = ST \left(I - \beta FT \right)^{-1} = \boldsymbol{\gamma}'$. ■

The key to understanding this result is that dispersed information removes the possibility of common knowledge. Since backward-looking solutions require co-ordination, and co-ordination requires common knowledge, the presence of dispersed information removes the possibility of backward-looking solutions.

For example, consider a candidate solution of the form $z_t = \boldsymbol{\delta}' X_t + \phi \overline{E}_t \left[z_{t-1} \right]$. When plugged into the equilibrium condition (3), this creates a term in $\overline{E}_t \left[\overline{E}_t \left[z_{t-1} \right] \right]$ that must correspond to $\overline{E}_t \left[z_{t-1} \right]$. If agents shared common knowledge, the law of iterated expectations would apply so that the second-order expectation collapses to the first-order. But without common knowledge, the two cannot be equal and the solution cannot hold.

Critically, since dispersed information ($\sigma_v > 0$) removes the very possibility of backward-looking solutions, the problem Blanchard and Kahn (1980) set out to solve is removed. Provided that the forward solution exists ($\beta\rho < 1$), no further parameter restrictions are required for uniqueness.

2.3 A comment

The toy model has a number of simplifications from a more general DSGE with imperfect common knowledge. In addition to being univariate, the equilibrium condition does not feature any lags of the endogenous variable (z_{t-1}) and agents' signals are exogenous, based only on x_t rather than (lags of) z_t. In the New Keynesian model considered below, all three of these are relaxed.

Implementing the solution in proposition 1 is also non-trivial. With an infinite state vector (X_t) it need not, in general, be possible to calculate the solution exactly and an approximation may need to be taken. For example, Nimark (2017) demonstrates that under quite general conditions an arbitrarily accurate approximation may be obtained simply by truncating the number of higher-order expectations at a pre-chosen cutoff k^*.

In static settings with exogenous signals, past literature has been able to re-write the exact solution as a function of a weighted average of higher-order expectations rather than of the complete hierarchy.[19] In the solution below, I demonstrate that this is also possible in a dynamic setting provided that endogenous signals are observed with a lag.

3 The NK model with zero trend inflation

I now move on to the New Keynesian model, starting from the canonical three equation variant of Galí (2008), extended only to deny full information to price-setting firms. It is cashless, and features Ricardian equivalence and lump sum taxes to eliminate any influence of fiscal policy. There is a continuum of firms, indexed $j \in [0, 1]$, that supply differentiated goods to a representative household, who values them via a Dixit-Stiglitz aggregator. The household provides labour to the firms in a competitive labour market. There is no capital. Firms are subject to Calvo-Yun pricing and information frictions, while the household and the central bank each possess full information. All agents are fully rational and trend inflation is taken to be zero. A derivation is provided in appendix C.

Combined with market clearing, the household's Euler equation is:

[19] See, for example, Woodford (2003*b*), Angeletos and La'O (2013) or Melosi (2014).

$$y_t = E_t^{\Omega}\left[y_{t+1}\right] - \sigma\left(i_t - \left(E_t^{\Omega}\left[p_{t+1}\right] - p_t\right) - x_t\right) \tag{30}$$

$$x_t = \rho x_{t-1} + u_t \tag{31}$$

where y_t is output; p_t is the aggregate price level; i_t is the nominal interest rate; σ is the elasticity of intertemporal substitution; x_t is a persistent demand shock (with $\rho \in (0,1)$ and $u_t \sim N\left(0, \sigma_u^2\right)$), implemented here as a shock to the natural rate of interest; and $E_t^{\Omega}\left[\cdot\right] = E\left[\cdot|\Omega_t\right]$ is the mathematical expectation conditional on all period-t information. The central bank makes use of a contemporaneous Taylor rule:

$$i_t = \phi_y y_t + \phi_\pi\left(p_t - p_{t-1}\right) \tag{32}$$

Individual firms have an independent probability, θ, of not being able to update their price in each period, so that the aggregate price level evolves as:

$$p_t = \theta p_{t-1} + \left(1-\theta\right) q_t \tag{33}$$

where $q_t \equiv \int_0^1 q_t(j)dj$ is the average reset price in period t. Firms' individual reset prices are given by their expectations of the optimal reset price:

$$q_t(j) = E_t(j)[q_t^*] \tag{34}$$

$$q_t^* = \left(1-\beta\theta\right)\left(p_t + \omega y_t\right) + \left(\beta\theta\right) E_t^{\Omega}\left[q_{t+1}^*\right] \tag{35}$$

where β is the household discount factor, ω is a function of the various elasticities of intertemporal substitution, demand, labour supply and marginal cost (so that ωy_t is real marginal cost); and $E_t(j)[\cdot] \equiv E\left[\cdot|\mathcal{I}_t\left(j\right)\right]$ is firm j's (rational) expectation based on an incomplete information set: $\mathcal{I}_t\left(j\right) \subset \Omega_t$. I show in the appendix that these together imply the following expression for the price level:

$$p_t = \theta p_{t-1} + \left(1 - \theta\left(1+\beta\right)\right)\overline{E}_t\left[p_t\right] + \left(\beta\theta\right)\overline{E}_t\left[p_{t+1}\right] + \left(1-\theta\right)\left(1-\beta\theta\right)\omega\overline{E}_t\left[y_t\right] \tag{36}$$

where $\overline{E}_t\left[\cdot\right] \equiv \int_0^1 E_t(j)[\cdot]\,dj$ is the average firm expectation. For reference, note that this may be readily rearranged (using $\pi_t \equiv p_t - p_{t-1}$) to give:

$$\begin{aligned}\pi_t = \left(1-\theta\right)\overline{E}_t\left[\pi_t\right] &+ \left(1-\theta\right)\left\{\overline{E}_t\left[p_{t-1}\right] - p_{t-1}\right\} \\ &+ \left(1-\theta\right)\left(1-\beta\theta\right)\omega\overline{E}_t\left[y_t\right] \\ &+ \left(\beta\theta\right)\overline{E}_t\left[\pi_{t+1}\right]\end{aligned} \tag{37}$$

which is the *Incomplete Information New Keynesian Phillips Curve*, first presented by Nimark (2008), although generalised here to allow for uncertainty about the previous period's price-level. It should be clear that with full information, period-t-dated expectations become accurate and the term in $\left\{\overline{E}_t\left[p_{t-1}\right] - p_{t-1}\right\}$ drops out, leading to the canonical *full information NKPC*:

$$\pi_t = \kappa y_t + \beta E_t^{\Omega}\left[\pi_{t+1}\right] \quad \text{where} \quad \kappa = \frac{\left(1-\theta\right)\left(1-\beta\theta\right)}{\theta}\omega \tag{38}$$

3.1 Firms' information

Firms retain complete information about the trend path for the economy, but have only incomplete and heterogeneous access to information about its deviations from that trend. Each period, each firm (regardless of whether they are free to adjust their price) observes a set of signals about the aggregate economy and uses these to update their beliefs. Note from equations (34)-(35) that there is strategic complementarity in firms' decision-making, so that each of them will care about not only the real marginal cost they will individually face but also the decisions (and, hence, beliefs) of all other firms.

As may already be clear, and will in any case be shown below, the underlying state of the economy includes the exogenous driving process and the lagged price level: $\boldsymbol{\eta}_t \equiv \begin{bmatrix} x_t & p_{t-1} \end{bmatrix}'$. I therefore assume that each firm's information set evolves as:

$$\boldsymbol{s}_t(j) = \boldsymbol{\eta}_t + \boldsymbol{v}_t(j) \quad \text{where} \quad \boldsymbol{v}_t(j) \sim N\left(\mathbf{0}, \sigma_v^2 I_2\right) \tag{39a}$$

$$\mathcal{I}_t(j) = \{\mathcal{I}_{t-1}(j), \boldsymbol{s}_t(j)\} \tag{39b}$$

The vector $\boldsymbol{s}_t(j)$ is firm j's set of signals in period t. The idiosyncratic noise, which I assume to be transitory, may be thought of as firms' failure to directly observe a public signal or a misinterpretation of the same (perhaps instead getting only an impression from newspaper coverage); an error of judgement; or as the imperfect applicability of national public signals to the aggregation level most relevant to each firm (e.g. at an industry or sector level). Idiosyncratic noise shocks are independent from aggregate shocks and each other, so that $\text{Cov}(u_t, \boldsymbol{v}_s(j)) = 0 \quad \forall j, s, t$ and $\text{Cov}(\boldsymbol{v}_s(i), \boldsymbol{v}_t(j)) = 0 \quad \forall i, j, s, t$.

Note, in particular, that firms do not observe the past price level perfectly. This assumption will prove to be critical in ensuring uniqueness below, but given the constantly-evolving nature of official estimates of economic data, it seems to be quite a weak assumption.[20] It bears emphasising, too, that uniqueness will only require the presence of *any* amount of idiosyncratic noise, no matter how small.

This signal structure has the benefit of nesting full information as a special case by setting $\sigma_v^2 = 0$, but other information assumptions could be made. Common noise shocks could be added to capture the effect of measurement errors by national statistical agencies or 'animal spirits'.[21,22] Alternatively, the signal regarding the natural rate of interest could be replaced with a similarly noisy signal about the previous period's aggregate output

[20]For example, the BEA conducts both an annual revision of US data, typically focusing on the preceding three years, and a 'comprehensive revision' of data every five years, in which all time periods of published data can be altered (Kornfeld et al., 2008). The comprehensive review conducted in 2013, for example, included changes to national accounts dating to 1929 (McCulla, Holdren and Smith, 2013).

[21]The main uniqueness result of the paper would still require that all signals include some (arbitrarily small) measure of idiosyncratic noise.

[22]A second way of capturing movements in sentiments, as described by Angeletos and La'O (2013), would be to grant firms noisy signals about other firms' signals. In either scenario, these would then be added, alongside the natural rate of interest, to the list of states that firms would need to estimate.

(e.g. each firm's own lagged marginal cost). This might arguably be a more plausible description of information actually used by firms in their pricing decision, but would no longer nest the case of full information. In the language of Baxter, Graham and Wright (2011), the model would then be only asymptotically invertible when $\sigma_v = 0$, rather than instantly invertible (the solution derived below would still apply when $\sigma_v > 0$).[23] Regardless, the uniqueness results will go through so long as all signals observed by firms include at least some idiosyncratic noise, in order to avoid common knowledge.

4 Solving the model

4.1 The purely forward-looking solution with full information

Before solving the model under ICK, I first solve it under full information. I also solve it in terms of the price level instead of inflation, as is more common. Defining $\delta \equiv 1/(1+\sigma\phi_y)$, imposing full information on the price-level (36) and combining it with (30) and (32), the model may be written compactly as:

$$A_0\boldsymbol{\zeta}_t = A_1 E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right] + B_1\boldsymbol{\zeta}_{t-1} + C_0 x_t \tag{40}$$

where $\boldsymbol{\zeta}_t = \begin{bmatrix} p_t & y_t \end{bmatrix}'$ and A_0, A_1, B_1 and C_0 are matrices of parameters.[24] The standard approach to solving models like (40) is to stack the variables and to rearrange it so that the forecast variables are on the left-hand side:[25,26]

$$\begin{bmatrix} E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right] \\ \boldsymbol{\zeta}_t \end{bmatrix} = \underbrace{\begin{bmatrix} A_1^{-1}A_0 & -A_1^{-1}B_1 \\ I & \mathbf{0} \end{bmatrix}}_{D} \begin{bmatrix} \boldsymbol{\zeta}_t \\ \boldsymbol{\zeta}_{t-1} \end{bmatrix} + \begin{bmatrix} -A_1^{-1}C_0 \\ \mathbf{0} \end{bmatrix} x_t \tag{41}$$

It is straightforward to show that, in this instance, D has four distinct eigenvalues:

$$\lambda \in \left\{ 0, 1, \frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} \pm \frac{\sqrt{\left(\beta+\delta+\kappa\sigma\delta\right)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)}}{2\beta\delta} \right\} \tag{42}$$

These are plotted below in figure 1.[27] Note, in particular, that $\frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} > 1$ and that the lower of the two quadratic solutions crosses $\lambda = 1$ when $\phi_\pi = 1 - \left(\frac{1-\beta}{\kappa}\right)\phi_y$ (the Taylor

[23] Another possibility could be *common (but incomplete) information*, in which all agents observe the same signal (sharing the same noise) – a setting explored, for example, by Levine et al. (2012). This adds additional dynamics to the model, as past noise shocks would affect current behaviour, but it would not address the question of determinacy. An equivalent multiplicity of solutions still emerges and the same equilibrium-selection assumptions are necessary as in the full information case.

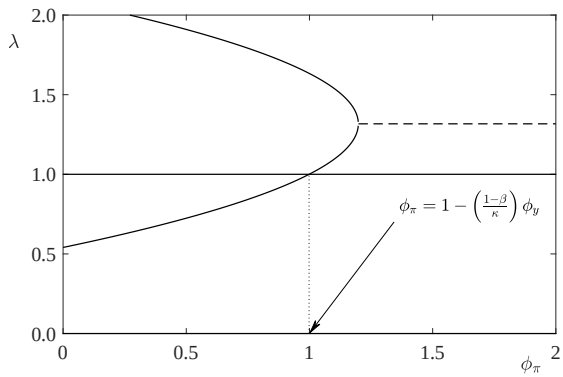
[24] $A_0 = \begin{bmatrix} \theta\left(1+\beta\right) & -\kappa\theta \\ \sigma\left(\phi_\pi+1\right) & \frac{1}{\delta} \end{bmatrix}$, $A_1 = \begin{bmatrix} \beta\theta & 0 \\ \sigma & 1 \end{bmatrix}$, $B_1 = \begin{bmatrix} \theta & 0 \\ \sigma\phi_\pi & 0 \end{bmatrix}$ and $C_0 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}$.

[25] The shock x_t may also be added to the stacked variables so that the driving process is i.i.d., but this would simply add ρ to the list of eigenvalues of D.

[26] If A_1 were not invertible, the generalized Schur form could be used, as per Klein (2000).

[27] The quadratic roots are complex when $\phi_\pi > \left(\frac{(1+\beta+\kappa\sigma)^2-4\beta}{4\beta\kappa\sigma}\right) - \left(\frac{1-\beta-\kappa\sigma}{2\kappa}\right)\phi_y + \left(\frac{\beta\sigma}{4\kappa}\right)\phi_y^2$.

threshold). When ϕ_π is above the Taylor threshold, the number of eigenvalues outside the unit circle matches the number of forecast variables, thus ensuring that backward-looking solutions will be explosive and the Blanchard-Kahn order condition is satisfied.



Note: The chart plots eigenvalues (λ) of the basic NK model when solved under full information as a function of the central bank's marginal response to inflation (ϕ_π). The dashed line represents the real component of two complex solutions. Structural parameters are $\{\beta, \phi_y, \sigma, \kappa\} = \left\{0.994, \frac{0.5}{4}, 1, 0.5\right\}$.

Figure 1: Eigenvalues of the New Keynesian model

To find the purely forward-looking solution, equation (40) can be substituted forward (see Cho and Moreno, 2011). Since the eigenvalues of the system are distinct, this is guaranteed to converge to the solution with the smallest eigenvalues in absolute value (see Rendahl, 2017).[28]

Proposition 4. *The purely forward-looking solution to the price level in* (31) *and* (40) *under full information is:*

$$p_t = \lambda\, p_{t-1} + \gamma\, x_t \tag{43a}$$

where

$$\lambda = \min\left\{1, \frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} - \sqrt{\left(\frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta}\right)^2 - \left(\frac{1+\kappa\sigma\delta\phi_\pi}{\beta\delta}\right)}\right\} \tag{43b}$$

$$\gamma = \frac{\kappa\sigma\delta}{(1-\delta\rho)\left(1+\kappa\sigma+\beta\left(1-\rho-\lambda\right)\right) - \kappa\sigma\frac{(1-\delta)(1-\delta\phi_\pi)}{(1-\delta\lambda)}} \tag{43c}$$

Proof. See appendix D.[29] ■

When $\phi_\pi > 1 - \left(\frac{1-\beta}{\kappa}\right)\phi_y$, the full information, purely forward-looking solution to the New Keynesian model has a unit root in prices. In this case, the more usual way

[28] McCallum (2007) refers to this as the 'MOD solution', meaning minimum-in-modulus.

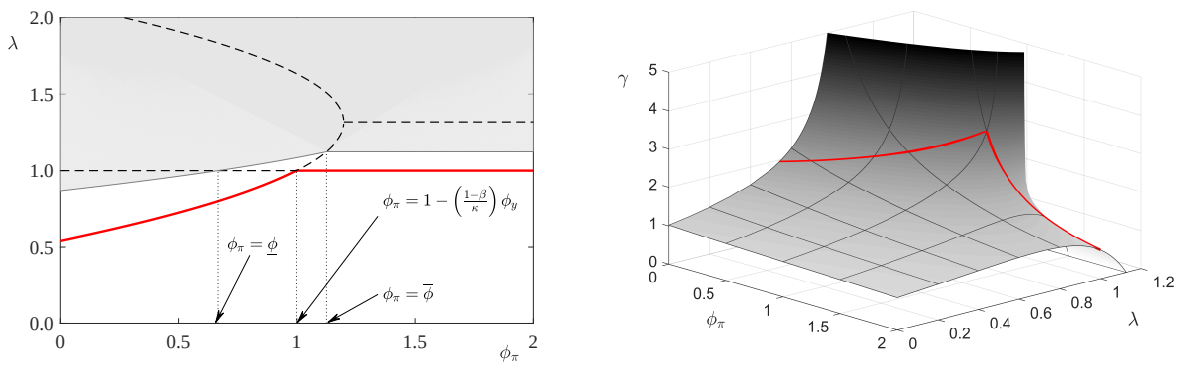
[29] Correspondingly, $y_t = \left(\frac{\sigma\left(\phi_\pi-(1+\phi_\pi)\lambda+\lambda^2\right)}{1+\sigma\phi_y-\lambda}\right)p_{t-1} + \left(\frac{\omega\gamma+\sigma(1-\gamma(1+\phi_\pi-\lambda-\rho))}{1+\sigma\phi_y-\rho}\right)x_t$. Note that the solution does not feature lagged demand (y_{t-1}) because it is not present in the Euler equation (as it would be if consumption habits were included, for example). This is also the reason for the eigenvalue of zero.

of writing the solution is in terms of inflation: $\pi_t = \gamma x_t$. When ϕ_π is below the Taylor threshold, however, the purely forward-looking solution features a stationary price level.

Lest readers be concerned with this stationarity, it bears noting that when x_t is sufficiently persistent, only this solution will produce a finite solution to γ.

Corollary 1. *A solution for λ* ***other than*** *that specified in (43b) would be economically plausible, in the sense that γ is positive and finite so that a positive demand shock raises prices, only when $\phi_\pi \in \left(\underline{\phi_\pi}, \overline{\phi_\pi}\right)$, where $\underline{\phi_\pi} = 1 - (1-\rho)\left(1 + \frac{1-\beta\rho}{\sigma\kappa}\right) - \left(\frac{1-\beta\rho}{\kappa}\right)\phi_y$ and $\overline{\phi_\pi} = 1 + \sigma\phi_y$. Furthermore, this interval vanishes as $\phi_y \to 0$ and $\rho \to 1$, with both $\underline{\phi_\pi}$ and $\overline{\phi_\pi}$ converging to 1.*

This point is illustrated in figure 2. Note that the region $\phi_\pi < \underline{\phi_\pi}$ with $\lambda = 1$ is the non-convergence region highlighted by Cho and McCallum (2015).



$$p_t = \lambda p_{t-1} + \gamma x_t$$

Note: The left-hand chart plots the solution coefficient λ (in red) against the eigenvalues of the system, with the grey shaded region covering values of λ for which γ would not be positive and finite: that is, such that a positive demand shock would fail to induce higher prices. The lower threshold is $\underline{\phi} = 1 - (1-\rho)\left(1 + \frac{1-\beta\rho}{\sigma\kappa}\right) - \left(\frac{1-\beta\rho}{\kappa}\right)\phi_y$, while the higher threshold is $\overline{\phi} = 1 + \sigma\phi_y$. The right-hand chart plots the solutions for γ that would emerge if λ and ϕ_π were both free parameters. Parameters are $\{\beta, \phi_y, \sigma, \kappa, \rho\} = \left\{0.994, \frac{0.5}{4}, 1, 0.5, 0.8\right\}$.

Figure 2: Economic plausibility of the New Keynesian model

4.2 The purely forward-looking solution under ICK

With firms making use of heterogeneous information sets, it becomes necessary to consider the infinite hierarchy of their (average) expectations ($X_t \equiv \left[\boldsymbol{\eta}_t' \quad \overline{E}_t\left[X_t\right]'\right]'$, where $\boldsymbol{\eta}_t \equiv \left[x_t \quad p_{t-1}\right]'$). Even when a solution can be mathematically described in terms of an infinite state vector, of course, it is not generally possible to calculate that solution. Instead, the solution may need to be estimated with a finite state that approximates the true, infinite state. In the case of ICK, Nimark (2017) shows that, under general condi-

tions, an arbitrarily accurate approximation may be found by truncating the hierarchy of expectations (this approach was first used in Nimark, 2008).

Until recently, the literature has generally held that when agents are forward-looking and observe endogenous signals (both of which apply in the NK model), a solution could *only* be expressed in terms of X_t. However, Huo and Takayama (2016) have demonstrated that an exact finite-state representation must exist, provided that agents do not observe endogenous signals *contemporaneously*. I show here that an exact finite-state representation may still be found when the endogenous signals are observed *with a lag*.

Let the 0th-order expectation of a variable be the variable itself; the 1st-order expectation be firms' average expectation about the variable; the 2nd-order expectation be firms' average expectation about the 1st-order expectation, and so on. Further, let $\widetilde{\boldsymbol{\eta}}_{t|t}$ be a geometrically-weighted average of firms' higher-order expectations regarding $\boldsymbol{\eta}_t$:

$$\widetilde{\boldsymbol{\eta}}_{t|t} \equiv (1-\varphi)\sum_{k=1}^{\infty}\varphi^{k-1}\boldsymbol{\eta}_{t|t}^{(k)} \quad \text{where} \quad \boldsymbol{\eta}_{t|t}^{(k)} \equiv \begin{cases} \boldsymbol{\eta}_t & \text{if } k=0 \\ \overline{E}_t\left[\boldsymbol{\eta}_{t|t}^{(k-1)}\right] & \text{if } k \geq 1 \end{cases} \tag{44}$$

for some $\varphi \in (-1,1)$. The parameter φ is the *equilibrium degree of strategic complementarity* in firms' price-setting, taking account of demand and the entire expected future path of prices.

With this definition in place, I am able to present the solution under ICK. The state of the economy follows a vector AR(1) process, inherited from the process for the exogenous shock and the recursive nature of the Kalman filter, while the price level is a function of the lagged price level (due to price stickiness) and weighted-average beliefs about the lagged price level and the current natural interest rate.

Proposition 5. *For the New Keynesian model with prices set under imperfect common knowledge, the state of the economy is given by:*

$$Z_t \equiv \begin{bmatrix} \boldsymbol{\eta}_t \\ \widetilde{\boldsymbol{\eta}}_{t|t} \end{bmatrix} = \begin{bmatrix} x_t & p_{t-1} & \widetilde{x}_{t|t} & \widetilde{p}_{t-1|t} \end{bmatrix}' \tag{45a}$$

and the purely forward-looking solution is of the form:

$$Z_t = AZ_{t-1} + Bu_t \tag{45b}$$

$$p_t = \theta\, p_{t-1} + (\lambda - \theta)\,\widetilde{p}_{t-1|t} + \gamma\,\widetilde{x}_{t|t} \tag{45c}$$

Furthermore, (45) equals the corresponding solution under full information (43) when $\sigma_v^2 = 0$ and approaches it smoothly as $\sigma_v^2 \to 0$.

Proof. See appendix E. ■

4.3 Uniqueness

Proposition 5 established the purely forward-looking solution under imperfect common knowledge, but it remains to demonstrate that ICK sufficies to rule out backward-looking solutions and rational bubbles – that is, to eliminate the possibility of sunspots. To begin, recall from equation (40) that under full information, the model may be written as:

$$A_0\boldsymbol{\zeta}_t = A_1 E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right] + B_1\boldsymbol{\zeta}_{t-1} + C_0 x_t \tag{46}$$

where $\boldsymbol{\zeta}_t \equiv \begin{bmatrix} p_t & y_t \end{bmatrix}'$. The purely forward-looking solution will be of the form:

$$\boldsymbol{\zeta}_t^{(F)} = \Lambda\boldsymbol{\zeta}_{t-1} + \Gamma x_t \tag{47a}$$

and the purely backward-looking solution is obtained by removing the expectation operator from $E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+1}\right]$ and rearranging:

$$\boldsymbol{\zeta}_t^{(B)} = A_1^{-1}A_0\boldsymbol{\zeta}_{t-1} - A_1^{-1}B_1\boldsymbol{\zeta}_{t-2} - A_1^{-1}C_0 x_{t-1} \tag{47b}$$

Furthermore, rational bubbles in this instance are any stochastic process, $\boldsymbol{\epsilon}_t$, that is orthogonal to x_t and satisfies equation (46) with the exogenous term (x_t) removed:

$$A_0\boldsymbol{\epsilon}_t = A_1 E_t^{\Omega}\left[\boldsymbol{\epsilon}_{t+1}\right] + B_1\boldsymbol{\epsilon}_{t-1} \tag{47c}$$

Following Blanchard (1979), the full set of solutions to (46) is then given by:

$$\boldsymbol{\zeta}_t = (1-\xi_t)\,\boldsymbol{\zeta}_t^{(F)} + \xi_t\boldsymbol{\zeta}_t^{(B)} + \boldsymbol{\epsilon}_t \quad \text{where} \quad \xi \in \mathbb{R} \tag{47d}$$

The elements ξ_t and $\boldsymbol{\epsilon}_t$ are both referred to as sunspot shocks – terms extrinsic to the model that nevertheless enter into the (full set of) solution(s).[30] The eigenvalue condition of Blanchard and Kahn (1980) then serves to ensure that $\boldsymbol{\zeta}_t^{(B)}$ and $\boldsymbol{\epsilon}_t$ are explosive. When combined with a further condition that the solution be stationary, the only admissible solution remaining has $\xi_t = 0$ and $\boldsymbol{\epsilon}_t = \mathbf{0}\,\forall t$ – that is, the purely forward-looking solution.

By contrast, when agents in the model face imperfect common knowledge, the purely backward-looking solution and the rational bubble simply do not exist, so any parameter restrictions that would render them explosive are therefore irrelevant. I address each type of sunspot in turn.

4.3.1 Rational bubbles

The rejection of rational bubbles proceeds equivalently to that for the toy model of section 2. For a rational bubble to feature in the solution, it must be perfectly observed by everybody.[31]

[30] For an example of a model that focuses on ξ as a sunspot, see Ascari, Bonomolo and Lopes (2016). For an example of a model that focuses on $\boldsymbol{\epsilon}_t$, see Flood and Garber (1980).

[31] Strictly, with a finite number of agents/firms, rejection of bubbles would require that at least two agents observe them with idiosyncratic noise in order to produce a hierarchy of expectations.

Proposition 6. *If price-setting firms in the NK model face imperfect common knowledge and any positive measure of firms, no matter how small, observes a candidate bubble $\boldsymbol{\epsilon}_t$ with any amount of idiosyncratic noise, no matter how small, then $\boldsymbol{\epsilon}_t$ cannot feature in a rational solution to the model.*

Proof. See appendix F.1. ■

4.3.2 Backward-looking solutions

Key to understanding the elimination of backward-looking solutions is the fact that such solutions require co-ordination between firms and co-ordination requires common knowledge. So long as firms' signals contain *any* amount of idiosyncratic noise, they can never perfectly agree on past values of state variables, so that co-ordination is not possible and backward-looking solutions are eliminated.

To help illustrate this point, before presenting the proposition in full I sketch a rejection of one specific candidate solution:

$$p_t = \boldsymbol{d}_p' Z_t + \mu p_{t-1} \tag{48}$$

This represents candidate solutions in which additional (if $\mu > 0$) weight is given to the lagged price level. Substituting this candidate into the equilibrium conditions of the model will produce an extra term in $\overline{E}_t\left[p_{t-1}\right]$ and, ultimately, a requirement that:

$$\overline{E}_t\left[p_{t-1}\right] = \chi\, p_{t-1} \tag{49}$$

for some coefficient χ, which must hold for (48) to be valid. But (49) is inconsistent with rational expectations. To see this, consider an individual firm's filter regarding p_{t-1}:

$$E_t(j)[p_{t-1}] = E_{t-1}(j)[p_{t-1}] + K_t\left\{\boldsymbol{s}_t\left(j\right) - E_{t-1}(j)[\boldsymbol{s}_t\left(j\right)]\right\} \tag{50}$$

for some projection matrix K_t. Taking the average of this and splitting out the two signals gives:

$$\begin{aligned}\overline{E}_t\left[p_{t-1}\right] = \overline{E}_{t-1}\left[p_{t-1}\right] + \rho\, K_{x,t}\left\{x_{t-1} - \overline{E}_{t-1}\left[x_{t-1}\right]\right\} + K_{x,t}\, u_t \\ + \quad K_{p,t}\left\{p_{t-1} - \overline{E}_{t-1}\left[p_{t-1}\right]\right\}\end{aligned} \tag{51}$$

Since u_t is unforecastable, p_{t-1} cannot be a function of it. A necessary condition for (49) to hold is therefore that $K_{x,t} = 0$. But since shocks are persistent ($\rho > 0$), this can only hold if (i) signals provide *no* information about the state ($\sigma_v = \infty$) or (ii) signals provide *full* information about the state ($\sigma_v = 0$).

Rejecting all backward-looking solutions

The full set of solutions featuring only the underlying fundamental (that is, those that exclude rational bubbles) must be expressible as:

$$\boldsymbol{\zeta}_t = \boldsymbol{a}\left(L\right) u_t \tag{52}$$

This captures any linear function of any current or past values of u_t. It therefore necessarily includes past values of $\boldsymbol{\zeta}_t$ and any average expectation, of any order and formed at any time, about any variable in the past. For the proof, however, I instead look for a solution in terms of:

$$Y_t \equiv \begin{bmatrix} \boldsymbol{\eta}_t \\ \overline{E}_t\left[Y_t\right] \end{bmatrix} \quad \text{where} \quad \boldsymbol{\eta}_t \equiv \begin{bmatrix} x_t & \boldsymbol{\zeta}'_{t-1} & \dots & x_{t-H} & \boldsymbol{\zeta}'_{t-H-1} \end{bmatrix}' \tag{53a}$$

$$Y_t = FY_{t-1} + Gu_t \tag{53b}$$

$$\underbrace{\begin{bmatrix} p_t \\ y_t \end{bmatrix}}_{\zeta_t} = \underbrace{\begin{bmatrix} \boldsymbol{d}'_p \\ \boldsymbol{d}'_y \end{bmatrix}}_{D} Y_t \tag{53c}$$

and let $H \to \infty$. Since $u_t = x_t - \rho x_{t-1}$, it should be clear that (52) and (53) are equivalent: for a given set of D coefficients, the corresponding $\boldsymbol{a}$ coefficients can be recovered and *vice versa*. To begin, I partition D (53c) as follows:

$$D = \Big[\underbrace{D^{(0)}_{x,0} \quad D^{(0)}_{\zeta,0} \quad \cdots \quad D^{(0)}_{x,H} \quad D^{(0)}_{\zeta,H}}_{\text{Sub-block (0)}} \Big| \underbrace{D^{(1)}_{x,0} \quad D^{(1)}_{\zeta,0} \quad \cdots \quad D^{(1)}_{x,H} \quad D^{(1)}_{\zeta,H}}_{\text{Sub-block (1)}} \Big| \cdots \Big] \tag{54}$$

where $D^{(k)}_{x,s}$ and $D^{(k)}_{\zeta,s}$ are the solution coefficients against the kth-order expectations about x_{t-s} and $\boldsymbol{\zeta}_{t-s-1}$ respectively. Since the model has one exogenous variable (x_t) and two endogenous variables (p_t, y_t), each sub-block of D is of size $(2 \times 3\left(1+H\right))$.

With this specification, it follows that a backward-looking solution must have either $D^{(k)}_{x,s} \neq 0$ or $D^{(k)}_{\zeta,s} \neq 0$ (or both) for some $s \geq 1$. That is, some non-zero weight must be placed on lags greater than those in the equilibrium conditions of the model. Similarly, the purely forward-looking solution will only have non-zero elements in $D^{(k)}_{x,0}$ and $D^{(k)}_{\zeta,0}$.

Proposition 7. *If price-setting firms in the NK model face imperfect common knowledge, all backward-looking solutions are rejected — that is, $D^{(k)}_{x,s} = 0$ and $D^{(k)}_{p,s} = 0$ for all $k \geq 0$ and $s \geq 1$ — regardless of the coefficients of the central bank's decision rule.*

Proof. See appendix F.2. ■

5 Some implications

The ability to identify a unique solution to an otherwise-standard New Keynesian model when the central bank does not satisfy the Taylor principle has a variety of implications for

how the model may be interpreted. I explore some of the most striking here, emphasising in advance that all are conditional on the model at hand, including the assumed common knowledge trend in prices.

It bears noting, first, that the unique solution under ICK is simply a pertubation from the purely forward-looking solution under full information. Indeed, since the solution under ICK approaches the forward solution under full information smoothly as firms' idiosyncratic noise goes to zero, the full-information forward solution may be used as a benchmark when considering the model's dynamics. In this respect, the addition of ICK may be thought of as an equilibrium selection device among full information solutions.

5.1 Impulse responses

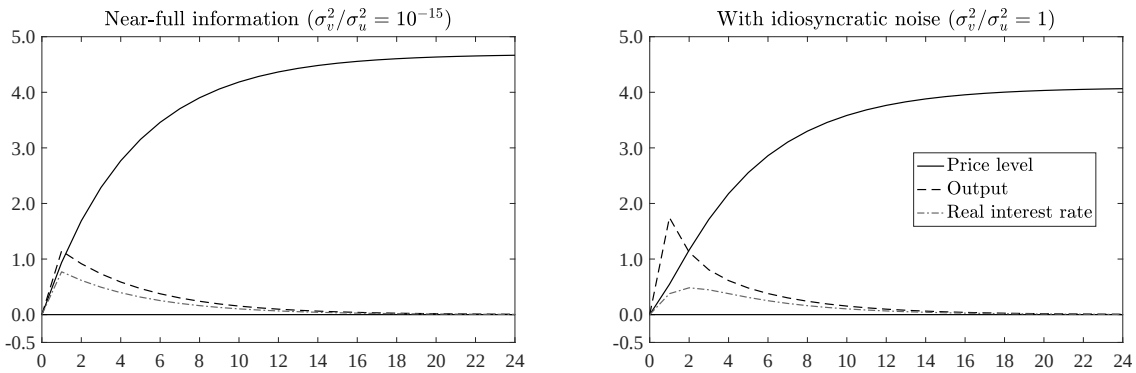
As a point of context for the corollaries listed below, figure 3 first provides impulse responses for the price level, output and the ex ante real interest rate following a positive shock to demand for different central bank designs and different levels of idiosyncratic noise. The left-hand panels plot those under near-full information, with $\sigma_v^2 = 10^{-15}$, while the right-hand panels plot those under idiosyncratically noisy information, with $\sigma_v^2 = 1$.

The top row implements a standard Taylor-type rule, with $\phi_\pi = 1.5$ and $\phi_y = 0.1$. The top-left panel therefore reproduces the results of the textbook New Keynesian model. The top-right panel plots responses when firms' signals have material amounts of idiosyncratic noise.[32] Even under the optimal signal extraction process, firms' beliefs are slow to update and prices consequently deviate by less than they do under full information. The reduced price response subsequently induces a larger response in output.

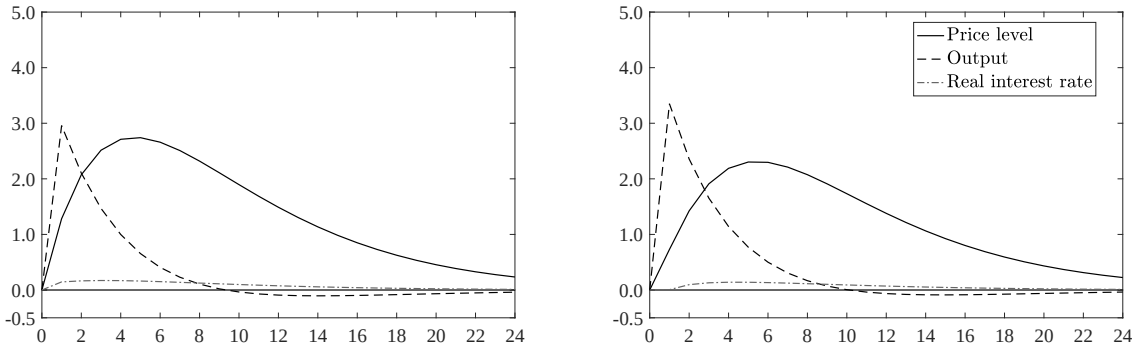
The middle row depicts the unique solutions (again, under near-full and dispersed information) when the central bank's marginal response to inflation is more subdued, at only 0.5 instead of 1.5. Since this coefficient is below the Taylor threshold, the aggregate price level itself becomes stationary, with inflation initially rising above trend and then falling below trend. The weaker price effect induces a larger movement in output on impact, but the sustained period of below-trend inflation later causes a small contraction. Despite the central bank's decision rule, the real interest rate remains above trend throughout because of the contribution of the period of below-trend inflation.

The bottom two panels show the unique solutions when the central bank does not respond to the state of the economy at all, instead keeping the nominal interest rate pegged at its steady-state level. The price response is both smaller and less persistent, causing the response of output to be substantially larger again. With no movement in the nominal interest rate, the real rate initially falls as the household anticipates the early above-trend inflation. Once the price level peaks and inflation falls below trend, however, the real interest rate rises, and remains, above trend thereafter.

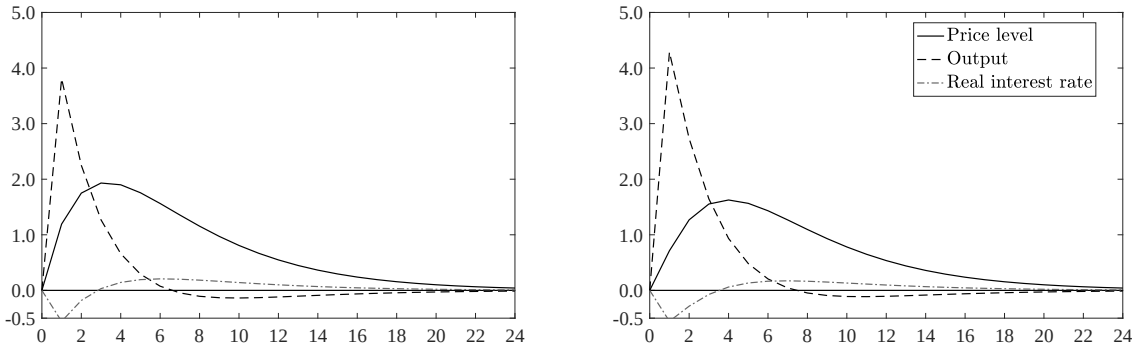
[32] It is therefore similar to Nimark (2008), albeit without firms having perfect sight of lagged prices.



(a) Standard Taylor rule ($\phi_\pi = 1.5$ and $\phi_y = 0.1$): $\lambda = 1.00$ $\gamma = 0.85$



(b) Subdued rule ($\phi_\pi = 0.5$ and $\phi_y = 0.1$): $\lambda = 0.84$ $\gamma = 1.09$



(c) State-invariant rule ($\phi_\pi = 0$ and $\phi_y = 0$): $\lambda = 0.70$ $\gamma = 1.03$

Note: The charts plot impulse response functions (IRFs) for the price level, output and the ex ante real interest rate following a positive shock to demand, when solutions for the price level under full information are: $p_t = \lambda p_{t-1} + \gamma x_t$. The left-hand panels impose near-full information ($\sigma_v^2 = 10^{-15}$), while in the right-hand panels firms' signals are subject to idiosyncratic noise ($\sigma_v^2 = 1$). Other parameters are $\{\beta, \sigma, \theta, \omega, \rho\} = \{0.994, 1, 2/3, 0.994, 0.8\}$.

Figure 3: Impulse responses following a demand shock

5.2 Central bank design determines price level persistence

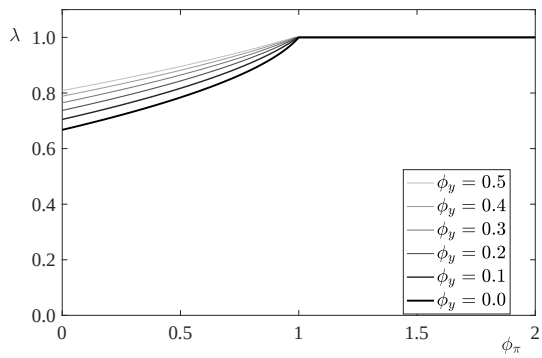
Corollary 2. *When the central bank chooses to satisfy the Taylor principle, the price level exhibits a unit root. When the central bank declines to satisfy the Taylor principle, the price level is stationary, with persistence strictly increasing in the coefficients of the*

central bank's decision rule:

$$\frac{\partial \lambda}{\partial \phi_\pi} = \kappa\sigma\delta\left((\beta+\delta+\kappa\sigma\delta)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)\right)^{-\frac{1}{2}} \qquad > 0 \tag{55a}$$

$$\frac{\partial \lambda}{\partial \phi_y} = \frac{1}{2}\sigma\left(1+\delta\left(2-\beta\right)\left((\beta+\delta+\kappa\sigma\delta)^2 - 4\beta\delta\left(1+\kappa\sigma\delta\phi_\pi\right)\right)^{-\frac{1}{2}}\right) > 0 \tag{55b}$$

Figure 4 plots λ as a function of ϕ_π while varying ϕ_y. The positive slope when below the Taylor threshold may be understood by considering the Euler equation and monetary policy rule together. Increasing ϕ_π lowers the weight that firms place on beliefs about current prices, but increases the coefficient on beliefs about the lagged price level.



Note: The chart plots the intrinsic persistence of the price level (λ) as a function of the central bank's marginal response to inflation (ϕ_π) for various values of ϕ_y. Other parameters are $\{\beta, \sigma, \kappa, \theta\} = \{0.994, 1, 0.5, 2/3\}$.

Figure 4: Persistence of the price level, varying ϕ_y and ϕ_π

When the central bank does not satisfy the Taylor principle, the persistence of the price level (λ) is also increasing in ϕ_y. Inspection of equation (E.28) in the appendix helps to explain this result. Firms' strategic complementarity in equilibrium is increasing in ϕ_y. Without full information, firms therefore limit their initial response to a shock, waiting until they gain more confidence that other firms are updating their prices too.

5.3 An interest rate peg

Corollary 3. *Provided that $\sigma_v > 0$, a unique and stable solution exists when the nominal interest rate remains pegged at its steady-state value ($\phi_y = \phi_\pi = 0$), with the following corresponding full-information coefficients:*

$$\lambda^{peg} = \frac{(\beta+1+\kappa\sigma) - \sqrt{(\beta+1+\kappa\sigma)^2 - 4\beta}}{2\beta} \quad \xrightarrow{\theta\to 0} \quad 0 \tag{56a}$$

$$\gamma^{peg} = \left(\frac{1}{1-\rho}\right)\left(\frac{\kappa\sigma}{1+\beta\left(1-\rho-\lambda\right)+\kappa\sigma}\right) \quad \xrightarrow{\theta\to 0} \quad \frac{1}{1-\rho} \tag{56b}$$

This result stands in contrast to the indeterminacy result of Sargent and Wallace (1975), although it bears emphasising that the peg here is restricted to the steady-state level of the interest rate.

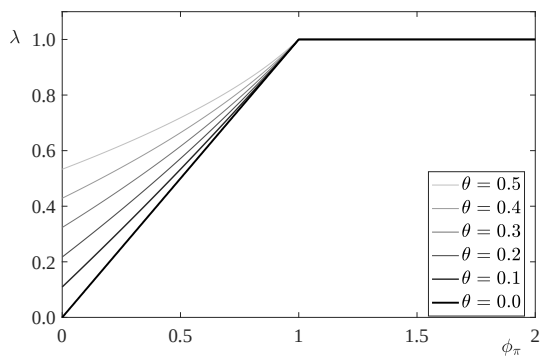
Note that under flexible prices ($\theta = 0$), these become simply $\lambda = 0$ and $\gamma = \frac{1}{1-\rho}$. This makes sense, as with an interest rate peg ($i_t = 0$) the household's Euler equation (30) becomes: $y_t = E_t\left[y_{t+1}\right] + \sigma\left\{E_t\left[p_{t+1}\right] - p_t + x_t\right\}$. If firms had full information (and sunspots could be ignored), expected inflation would then adjust to offset x_t, keeping the term in braces equal to zero so that output never deviates from trend.

5.4 Price stickiness and aggregate price level persistence

Corollary 4. *When the central bank declines to satisfy the Taylor principle, the intrinsic persistence of the price level is strictly increasing in the stickiness of individual prices:*

$$\frac{\partial \lambda}{\partial \theta} = \frac{\sigma\delta\left(\kappa + \omega\left(1+\beta\right)\right)\left(\lambda - \phi_\pi\right)}{\theta\left(\left(\beta + \delta + \kappa\sigma\delta\right)^2 - 4\beta\delta\left(1 + \kappa\sigma\delta\phi_\pi\right)\right)^{\frac{1}{2}}} > 0 \tag{57}$$

Figure 5 highlights a curious oddity that has long applied to the canonical solution to the New Keynesian model. When the central bank satisfies the Taylor principle, so that $\lambda = 1$, changing the stickiness of firms' prices (θ) does not alter the persistence of the model following a shock, only the magnitude of its effect. By contrast, when the central bank does not satisfy the Taylor principle, increasing θ does instead achieve the intuitively anticipated result of increasing the model's endogenous persistence.



Note: The chart plots the intrinsic persistence of the price level (λ) as a function of the central bank's marginal response to inflation (ϕ_π) for various values of price stickiness (θ). Other parameters are $\{\beta, \sigma, \kappa, \phi_y\} = \{0.994, 1, 0.5, 0.1\}$.

Figure 5: Persistence of the price level, varying θ and ϕ_π

5.5 The real interest rate still responds

It is commonly suggested that the purpose of the Taylor principle is to ensure that the real interest rate moves in the same direction as prices (inflation). However, this is not

necessary when the price level is stationary. Following a demand shock that initially raises prices, the period of below-trend inflation that occurs to bring the price level back to trend will also raise the real interest rate, even if the nominal rate remains fixed.

Corollary 5. *Suppose that $\phi_y = 0$. Then under full information:*

(i) the ex ante real interest rate is given by:

$$r_t = (1 + \phi_\pi - \rho - \lambda)\,\gamma\, x_t + (1 - \lambda)\,(\lambda - \phi_\pi)\, p_{t-1} \tag{58a}$$

(ii) the impulse response function (IRF) of the real interest rate is given by:

$$\frac{\partial r_{t+s}}{\partial u_t} = \gamma \left((1 + \phi_\pi - \rho - \lambda)\,\rho^s + \frac{(1-\lambda)\,(\lambda - \phi_\pi)}{(\lambda - \rho)} \left(\lambda^s - \rho^s\right) \right) \text{ ; and} \tag{58b}$$

(iii) the sum of all current and future IRF values is given by:

$$\Xi_r \equiv \sum_{s=0}^{\infty} \frac{\partial r_{t+s}}{\partial u_t} = \begin{cases} \gamma & > 0 \quad \text{if } \phi_\pi \leq 1 \\ \gamma \left(\frac{\phi_\pi - \rho}{1-\rho}\right) & > 0 \quad \text{if } \phi_\pi > 1 \end{cases} \tag{58c}$$

When the Taylor principle is satisfied, the impulse response simplifies to $\frac{\partial r_{t+s}}{\partial u_t} = \gamma\,(\phi_\pi - \rho)\,\rho^s$, which is always positive. When the Taylor principle is not satisfied, the real rate will be negative on impact if $1 + \phi_\pi - \rho - \lambda < 0$ – that is, if the initial move in the nominal interest rate is unsufficient to offset the initial increase in prices. Even then, however, it eventually turns positive and the absolute sum of later periods exceeds that of early periods so that the total effect is positive.

As firms' signal noise increases, the sum over time of real interest rates falls (since more sluggish expectations imply smaller inflation deviations), but it remains strictly positive. Figure 6 illustrates this point, plotting Ξ_r for various values of ϕ_π as the amount of idiosyncratic noise varies. Although not shown, setting $\phi_y > 0$ raises Ξ_r in all cases.

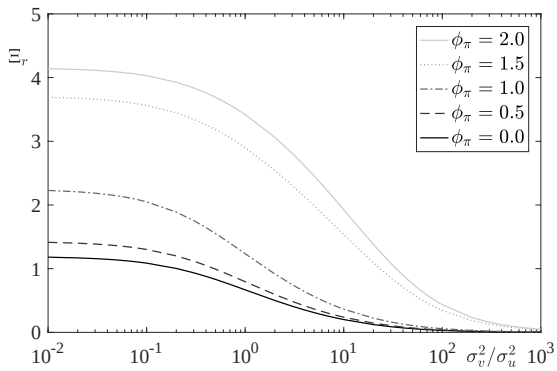
5.6 'Passive' monetary policy can still deliver stable inflation

Corollary 6. *With only demand shocks, the unconditional variance of inflation is:*

$$\mathrm{Var}\,(\pi_t) = 2\gamma^2 \left(\frac{1-\lambda}{1-\lambda^2}\right)\left(\frac{1-\rho}{1-\rho\lambda}\right) \mathrm{Var}\,(x_t) \tag{59}$$

under full information and strictly falls as σ_v rises.

Varying ϕ_π, the unconditional variance of inflation (59) peaks at the Taylor threshold. When $\phi_\pi > \phi_\pi^{\text{Taylor}}$, inflation volatility is decreasing in ϕ_π, as is well understood in the literature. In this case, $\lambda = 1$ so that (59) simplifies to $\mathrm{Var}\,(\pi_t) = \gamma^2\,\mathrm{Var}\,(x_t)$. An increase

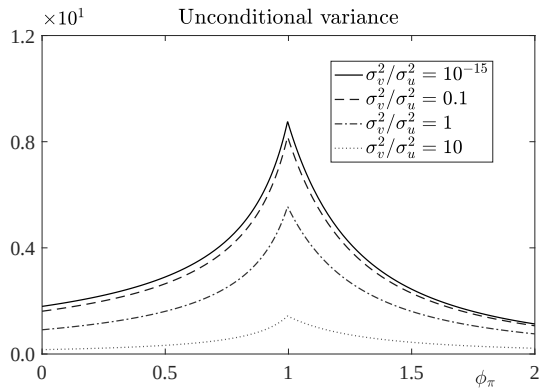


Note: The chart plots the sum of all current and future deviations of the real interest rate from trend caused by a positive demand shock ($\Xi_r \equiv \sum_{s=0}^{\infty} \frac{\partial r_{t+s}}{\partial u_t}$) as a function of the level of idiosyncratic noise faced by price-setting firms (σ_v^2/σ_u^2) for various values of the central bank's marginal response to inflation (ϕ_π). Other parameters are $\{\beta, \phi_y, \sigma, \theta, \omega, \rho\} = \{0.994, 0.1, 1, 2/3, 0.994, 0.8\}$.

Figure 6: The total effect of a demand shock on the real interest rate

in ϕ_π dampens the response of aggregate demand to shocks, thus lowering γ and, with it, inflation volatility.

When $\phi_\pi < \phi_\pi^{\text{Taylor}}$, however, there are two effects at play. First, the usual damping effect on demand applies, just as for values of $\phi_\pi > \phi_\pi^{\text{Taylor}}$. In addition to this, increasing ϕ_π also increases the persistence of the price level (λ). For all plausible calibrations, this second effect dominates, causing volatility to rise with ϕ_π. Above the Taylor threshold, λ no longer varies with ϕ_π so that the damping effect begins to dominate.

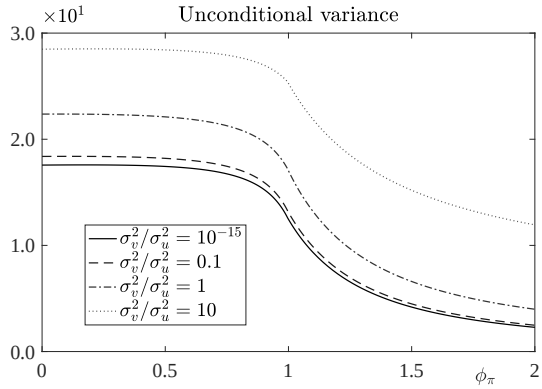


Note: The chart plots the unconditional variance of deviations of inflation from trend as a function of the central bank's marginal response to inflation (ϕ_π) for various amounts of idiosyncratic noise in firms' signals (σ_v^2/σ_u^2). Other parameters are $\{\phi_y, \beta, \sigma, \theta, \omega, \rho\} = \{0.1, 0.994, 1, 2/3, 0.994, 0.8\}$.

Figure 7: Unconditional volatility in inflation

Figure 7 illustrates this point, plotting the unconditional variance of inflation as a function of ϕ_π while holding $\phi_y = 0.1$. Increasing the amount of noise in firms' signals directly lowers inflation volatility as this causes their beliefs to update more slowly, leading to smaller price changes in each period.

Curiously, this peaked result for inflation volatility does not flow through to output volatility. Figure 8 plots the equivalent chart for the variance of output. When ϕ_π is above the Taylor threshold, output volatility is decreasing in ϕ_π through the same dampening mechanism as for inflation. When ϕ_π is below the Taylor threshold, however, output volatility remains high regardless of the value of ϕ_π.



Note: The chart plots the unconditional variance of deviations of output from trend as a function of the central bank's marginal response to inflation (ϕ_π) for various amounts of idiosyncratic noise in firms' signals (σ_v^2/σ_u^2). Other parameters are $\{\phi_y, \beta, \sigma, \theta, \omega, \rho\} = \{0.1, 0.994, 1, 2/3, 0.994, 0.8\}$.

Figure 8: Unconditional volatility in output

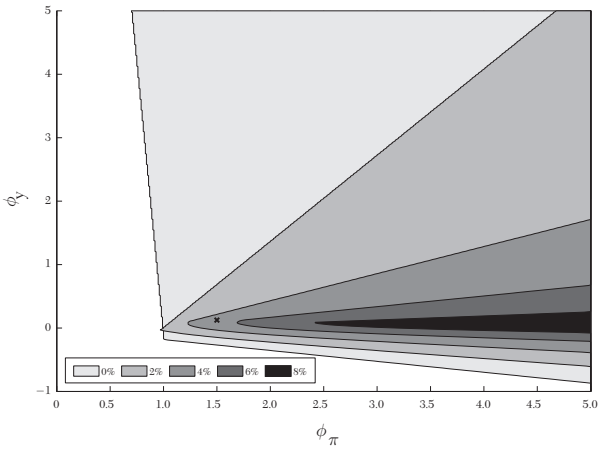
This comes from the innovation being a demand shock in a New Keyensian model, and may be understood by comparing the IRFs under a weak policy rule and an interest rate peg in figure 3. Following such a shock, adjustment must take place through either output or prices. When ϕ_π is very low, price movements are small and so the on-impact output movement is large. As ϕ_π is increased, the price response becomes larger and the on-impact output response correspondingly smaller. Against this, output returns to trend more quickly when ϕ_π is very low, despite the larger initial movement. Since the persistence in the demand deviation increases alongside its decrease initial response and the usual damping effect as ϕ_π rises, overall volatility remains largely unchanged.

6 Extension to a model with positive trend inflation

Several authors have noted[33] that the conditions for determinacy change in the presence of positive trend inflation and that, in particular, the determinacy region for parameters in a Taylor-type rule shrinks as trend inflation increases. This result, illustrated in figure 9, underlies one of the main criticisms of post-crisis proposals to raise inflation targets in order to reduce the probability of interest rates being restricted by their lower bounds.[34]

[33]See, for example, Ascari and Ropele (2009), Coibion and Gorodnichenko (2011) and Ascari and Sbordone (2014).

[34]For examples of arguments in favour of raising inflation targets, see Ball (2014) or Krugman (2014). For examples of criticisms of this proposal based, in part, on indeterminacy or the volatility that would



Note: The chart, which is reproduced from Ascari and Sbordone (2014), plots parameter regions of the central bank's decision rule that produce determinacy for different values of annual trend inflation in the New Keynesian model when solved under full information. The **x** marks the point where $\phi_\pi = 1.5$ and $\phi_y = 0.5/4$. Other parameters are $\{\alpha, \beta, \sigma, \psi, \varepsilon, \theta\} = \{0, 0.99, 1, 1, 10, 0.75\}$.

Figure 9: Determinacy in the NK model with trend inflation under full information

A key extension of the previous section is then to illustrate that this concern with determinacy in a model with full information is not robust to the introduction of information frictions on the part of price setters. In the appendix I provide a derivation of the basic New Keynesian model extended to include both non-zero trend inflation and imperfect common knowledge among firms. The log-linearised system emerges as:

$$x_t = \rho x_{t-1} + u_t \tag{60a}$$

$$y_t = E_t^{\Omega}\left[y_{t+1}\right] - \sigma\left(i_t - E_t^{\Omega}\left[p_{t+1}\right] + p_t - x_t\right) \tag{60b}$$

$$i_t = \phi_\pi\left(p_t - p_{t-1}\right) + \phi_y y_t \tag{60c}$$

$$p_t = \theta\overline{\Pi}^{\varepsilon-1} p_{t-1} + \left(1 - \theta\overline{\Pi}^{\varepsilon-1}\right)\left(\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}\right)\overline{E}_t\left[\psi_t - \phi_t\right] \tag{60d}$$

$$\psi_t = \left(1 - \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left((1+\omega)\, y_t + \frac{1}{\psi}s_t + \frac{\varepsilon}{1-\alpha}p_t\right) + \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\left(\psi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{60e}$$

$$\phi_t = \left(1 - \beta\theta\overline{\Pi}^{\varepsilon-1}\right)\left(y_t + (\varepsilon - 1)\, p_t\right) + \beta\theta\overline{\Pi}^{\varepsilon-1}\left(\phi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{60f}$$

$$s_t = \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right) s_{t-1} + \left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\left(p_t - p_{t-1}\right) \tag{60g}$$

where x_t is a persistent demand shock, p_t is the price level; y_t is aggregate demand (and output); i_t is the nominal interest rate; ψ_t is the average present discounted value of future marginal cost; ϕ_t is the same for future marginal revenue; and s_t is the level of price dispersion across firms. Among the parameters, α parameterises the curvature of the production function (in labour only); β is the household discount factor; σ is the EIS; ψ is the Frish elasticity of labour supply; ε is the elasticity of substitution between

ensue with sunspot shocks, see Bernanke (2010) or Ascari, Florio and Gobbi (2017).

differentiated goods; θ is the (Calvo) probability of a firm not updating its price in a given period; ϕ_π and ϕ_y are the usual parameters of the central bank's decision rule; and $\omega = \frac{1}{\sigma} - 1 + \left(\frac{1}{1-\alpha}\right)\left(1 + \frac{1}{\psi}\right)$. Finally, $\overline{\Pi}$ is the gross, quarterly trend rate of inflation.

Note that unlike most derivations of the NK model with trend inflation, ψ_t and ϕ_t are here written in nominal terms instead of real. A number of elements have also not been cancelled out between them because of the absence of full information. The introduction of trend inflation implies that price dispersion is non-zero in the linearised model, so the minimum state under full information is $\boldsymbol{\eta}_t \equiv \begin{bmatrix} x_t & p_{t-1} & s_{t-1} \end{bmatrix}'$. As before, I suppose that, each period, firms observe noisy signals of that underlying state:

$$\boldsymbol{s}_t(i) = \boldsymbol{\eta}_t + \boldsymbol{v}_t(i) \qquad \boldsymbol{v}_t(i) \sim N\left(\mathbf{0}, \sigma_v^2 I\right) \tag{61}$$

where $\boldsymbol{v}_t(i)$ is idiosyncratic white noise, independent both across firms and over time. This then nests full information by simply taking the variance of $\boldsymbol{v}_t(i)$ down to zero.

Proposition 8. *When a solution to (60)-(61) exists, that solution is unique, regardless of the values of ϕ_y and ϕ_π, and takes the form:*

$$p_t = \boldsymbol{\mu}_p' Z_t \quad , \quad y_t = \boldsymbol{\mu}_y' Z_t \quad , \quad \textit{etc.} \tag{62a}$$

$$Z_t = AZ_{t-1} + Bu_t \tag{62b}$$

where

$$Z_t \equiv \begin{bmatrix} \boldsymbol{\eta}_t \\ \widetilde{\boldsymbol{\eta}}_{t|t} \end{bmatrix} \tag{62c}$$

$$\widetilde{\boldsymbol{\eta}}_{t|t} \equiv (1-\varphi)\sum_{k=1}^{\infty} \varphi^{k-1} \boldsymbol{\eta}_{t|t}^{(k)} \tag{62d}$$

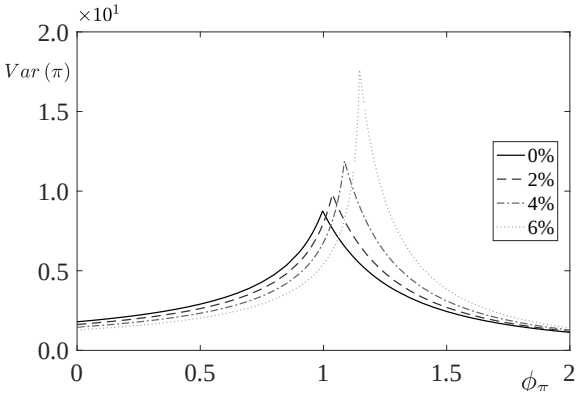
$$\boldsymbol{\eta}_{t|t}^{(k)} \equiv \begin{cases} \boldsymbol{\eta}_t & \textit{if } k = 0 \\ \overline{E}_t\left[\boldsymbol{\eta}_{t|t}^{(k-1)}\right] & \textit{if } k \geq 1 \end{cases} \tag{62e}$$

Furthermore, that solution equals the purely forward-looking solution under full information when $\sigma_v = 0$ and approaches it smoothly as $\sigma_v \to 0$.

Proof. See appendix G. ■

As an illustration of the results that emerge when the NK model with trend inflation is solved under ICK, figure 10 plots the unconditional variance of log deviations of inflation from trend in the model of (60) for different rates of annual trend inflation. The variance of firms' idiosyncratic noise has been set to $\sigma_v^2 = 10^{-15}$, so that the results are comparible to a setting of universal full information.

The solid black line corresponds to the same line in figure 7, with trend inflation at zero. As trend inflation increases, the peak in inflation volatility shifts to right. This



Note: The chart plots the unconditional variance of log-deviations of inflation from its trend, as a function of the central bank's marginal responds to those deviations (ϕ_π, for different values of annual trend inflation. Firms are assumed to have near-full information, with $\sigma_v^2 = 10^-15$. Other parameters are $\{\alpha, \beta, \sigma, \psi, \varepsilon, \theta\} = \{1/3, 0.994, 1, 1, 4, 2/3\}$.

Figure 10: Inflation volatility as a function of trend inflation in the NK model

corresponds to the value of ϕ_π above which the eigenvalue restriction of the Blanchard-Kahn conditions is satisfied in the underlying full information model, and so matches the left-most edge of each shaded area in figure 9.[35] To the right of each peak, raising ϕ_π has the usual effect of damping inflation volatility.

To the left of each peak, where a model with full information would have indeterminacy and sunspots, the NK model with ICK admits a unique solution in which inflation volatility is increasing in ϕ_π. The reason for this is unchanged from the case of zero trend inflation. In this region, the price level is stationary, but with intrinsic persistence that is increasing in the strategic complementarity faced by firms which is itself increasing in ϕ_π. The increase in volatility from this increasing persistence outweighs the direct damping effect of further activism by the central bank, causing overall variance to rise. The price level achieves a unit root at the peak shown and remains there as ϕ_π continues to rise, meaning that only the damping effect is further strengthened for higher ϕ_π.

All told, and conditional on the model at hand, raising the inflation target while keeping the central bank's decision rule unchanged *would* increase inflation volatility (assuming that the decision rule was of the usual Taylor variety), but it need not increase by overly much, and would not run the risk of extrinsic volatility affecting the economy. It bears emphasising, however, that the model as presented has been log-linearised around its respective trend, assuming that trend to be universally known. The full dynamics of a transition to a new steady state remain unexplored here, and may well require the forceful application of monetary policy to shape agents' beliefs regarding that trend.

[35] The two do not perfectly coincide, as other parameters differ between them.

7 Liquidity Traps

The previous sections focused on local determinacy in the log-linearised version of the NK model, implicitly assuming that the trend of the model was constant over time. In addition to local determinacy, however, the NK model famously has the possibility of multiple *steady-state* equilibria when the nominal interest rate is subject to a lower bound, as emphasised by Benhabib, Schmitt-Grohe and Uribe (2001, 2002).

When at trend, $x_t = 0 \, \forall t$, firms have full information, prices are flexible and output remains at its steady-state level. The non-linear equations of the model then become:

$$1 = \beta \left(1 + \bar{i}_t\right) \frac{1}{\overline{\Pi}_{t+1}} \tag{63}$$

$$1 + \bar{i}_t = \max \left\{ 1, \frac{\Pi^*}{\beta} \left(\frac{\overline{\Pi}_t}{\Pi^*} \right)^{\phi_\pi} \right\} \tag{64}$$

where the first line is the household Euler equation and the second is the central bank's decision rule, expanded to include a zero lower bound on the net nominal interest rate. Trend variables have a line over them, Π^* is the central bank's gross inflation target and $\frac{\Pi^*}{\beta}$ is the gross nominal interest rate when at the central bank's preferred equilibrium. Combining the two gives the resultant difference equation for trend inflation:

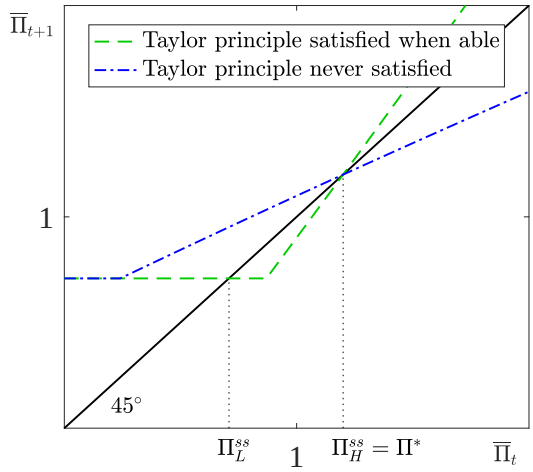
$$\overline{\Pi}_{t+1} = \beta \max \left\{ 1, \frac{\Pi^*}{\beta} \left(\frac{\overline{\Pi}_t}{\Pi^*} \right)^{\phi_\pi} \right\} \tag{65}$$

This is illustrated in figure 11. If the Taylor principle is adhered to where possible, two steady-state equilibria emerge – the desired one at the chosen inflation target and a deflationary one. Of these, the deflationary equilibrium is globally stable and the desired equilibrium is globally unstable. Because the deflationary equilibrium is globally stable, it is commonly described as a 'liquidity trap'.

When the central bank declines to satisfy the Taylor principle, however, the desired steady-state equilibrium is both unique and globally stable, even when the nominal interest rate is subject to a lower bound. If price-setting firms had full information regarding x_t and past deviations from the trend path of the price level, there would be local indeterminacy in the vicinity of this equilibrium. With ICK, however, local determinacy is obtained, thus granting full uniqueness for the entire non-linear model.

8 Conclusion

When price-setting firms are subject to idiosyncratic noise about both current and past deviations of the economy from its trend, the solution to the linearised NK model is unique (ruling out sunspots) and features nominal stability, regardless of the reaction



Note: The green dashed line charts the evolution of trend inflation when the central bank satisfies the Taylor principle when it is able ($\phi_\pi = 1.5$), subject to a (zero) lower bound on the nominal interest rate. The blue dot-dashed line charts the evolution when the central bank never satisfies the Taylor principle ($\phi_\pi = 0.5$).

Figure 11: Evolution of trend inflation in the New Keynesian model

of the central bank. Standard solutions to the New Keynesian model remain when the Taylor principle is satisfied and the noise faced by firms is taken to zero. But when the Taylor principle is not satisfied, including when the nominal interest rate is simply pegged to its steady-state level, a unique and stable solution emerges that features stationarity in the aggregate price level, provided that firms face at least some heterogeneous uncertainty. In all cases, as is typical in such models, the information friction represents a real rigidity, with persistence following a shock increasing in the amount of noise faced by firms.

The uniqueness result rests critically on agents never being able to perfectly agree on past states of the economy. In practice, this is a remarkably weak assumption. Rational bubbles are eliminated, for example, so long as at least two agents (in an economy of millions) are subject to even an epsilon of idiosyncratic noise when observing those bubbles,[36] so that average expectations do not satisfy the law of iterated expectations. This result extends to settings of positive trend inflation, removing the possibility of sunspot uncertainty if central banks were to raise their inflation targets, provided that the revised target was sufficiently well communicated as to shift beliefs about trend.

The determinacy obtained under an interest rate peg is striking, but ultimately perfectly intuitive. The peg applied above is to the steady-state value for the nominal interest rate (which, with trend inflation at zero, is just the steady state real interest rate, here $1/\beta$). So long as the natural interest rate returns to this value, and firms *know*

[36] 'At least two' among a finite number of agents is equivalent to 'a strictly positive measure' for a continuum of agents in this context, as both are the minimum required to give rise to average expectations that are not common knowledge.

that it will return, then the logic of Wicksell (1898) remains intact. If the interest rate were indefinitely pegged to a different value, however, then it would represent a change of trend itself. Dynamics would then depend on the extent to which agents recognise the interest rate peg as a change in the inflation target.

It is important to emphasise that the model, as implemented, is log-linearised around a deterministic steady state. This imposes an assumption that although firms do not share common knowledge about the actual price level, they perfectly observe (and agree on) its underlying trend. In effect, this amounts to an assumption that while firms' expectations about near-term inflation remain dispersed, their beliefs about long-run inflation are perfectly anchored. Conditional on this assumption, nominal stability around that trend need not require a systematic central bank response to the state of the economy.

Indeed, when firms have full information about the trend and the central bank's response to inflation is less than one, the full, non-linear model features a unique, globally stable steady-state equilibrium even after allowing for the possibility of a lower bound on interest rates (removing the deflationary trap emphasised by Benhabib, Schmitt-Grohe and Uribe (2001)). This paper makes no comment on how agents might arrive at consensus about the steady state of the economy if there were also uncertainty regarding trend. If, for example, systematic policy is necessary to ensure that long-run expectations remain well anchored then that would be in addition to the results discussed above.

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Appendix

A Estimating a Taylor rule for the Federal Reserve

All data are quarterly. Federal Funds Rate data and shadow rate data are taken from Wu and Xia (2016) – I use the interest rate at the end of the first month of each quarter. All other data are taken from the Federal Reserve Economic Data published by the Federal Reserve Bank of St. Louis. These include (data codes in parentheses): real GDP (GDPC1); real potential GDP (GDPPOT); the annualised quarterly change in the GDP deflator (A191RI1Q225SBEA); the annualised quarterly change in the producer price index for all commodities (PPIACO_PCA); the annualised quarterly change in the M2 money stock (M2SL_PCA); the quarterly average 10-year constant maturity treasury rate (GS10); and the quarterly average 3-month treasury bill rate (TB3MS).

The estimated equation is:

$$r_t = \rho\, r_{t-1} + (1-\rho)\,(\bar{r} + \phi_\pi \pi_t + \phi_y y_t) + \varepsilon_t \tag{A.1}$$

where r_t is the federal funds rate prior to the financial crisis of 2008 and the Wu and Xia (2016) shadow rate thereafter; π_t is inflation; and y_t is the output gap. I stop the sample at 2015q4 as that was the last quarter in which the federal funds target range was at its lower bound. For the GMM estimate, I use one lag of the interest rate, inflation rate, output gap, commodity inflation, money growth and treasury spread as instruments. Note that there are only 29 observations in the post-crisis period, so the GMM results for that period should be treated with caution.

Period	OLS			GMM			Std. dev.
	ϕ_π	ϕ_y	ρ	ϕ_π	ϕ_y	ρ	of inflation
1960:1 – 1979:2	0.80	0.72	0.73	0.81	0.61	0.66	2.83
	(0.14)	(0.25)	(0.08)	(0.12)	(0.19)	(0.08)	
1979:3 – 2008:2	2.09	1.45	0.88	2.52	0.83	0.83	2.07
	(0.45)	(0.67)	(0.04)	(0.63)	(0.38)	(0.04)	
2008:3 – 2015:4	-0.07	0.20	0.83	-2.39	0.53	0.87	0.86
	(0.55)	(0.40)	(0.08)	(2.96)	(1.10)	(0.14)	

Table A.1: Contemporaneous Taylor rules

To allow a more direct comparison to Clarida, Galí and Gertler (2000), I also estimate the same for a forward-looking rule, replacing π_t and y_t with π_{t+1} and y_{t+1} respectively (so that ε_t includes forecast errors):

Period	OLS			GMM			Std. dev.
	ϕ_π	ϕ_y	ρ	ϕ_π	ϕ_y	ρ	of inflation
1960:1 – 1979:2	0.94	0.50	0.70	0.89	0.76	0.76	2.83
	(0.13)	(0.19)	(0.07)	(0.18)	(0.34)	(0.08)	
1979:3 – 2008:2	2.52	2.60	0.91	2.90	1.01	0.85	2.07
	(0.86)	(1.54)	(0.04)	(0.73)	(0.48)	(0.04)	
2008:3 – 2015:4	-0.43	-0.05	0.78	-2.39	-0.69	-0.14	0.86
	(0.44)	(0.34)	(0.11)	(0.79)	(0.35)	(0.12)	

Table A.2: Forward-looking Taylor rules

B Solving the toy model: proof of proposition 1

B.1 The reduced-form expression for z_t.

Substituting (10) into (3), I obtain:

$$\boldsymbol{\gamma}' X_t = \beta \boldsymbol{\gamma}' \overline{E}_t \left[F X_t + G u_{t+1} \right] + \overline{E}_t \left[x_t \right] \tag{B.1}$$

Recall that S and T are selection matrices such that $SX_t = x_t$ and $TX_t = \overline{E}_t \left[X_t \right]$. Then after straightforward manipulation this becomes:

$$\boldsymbol{\gamma}' = ST \left(\sum_{s=0}^{\infty} (\beta F T)^s \right) \tag{B.2}$$

Since T is a shift operator, this sum will be finite only if the spectral radius of βF is less than one. Since the largest eigenvalue of F is ρ (see below), the spectral radius is less than one when $\beta\rho < 1$. In this case, the solution becomes:

$$\boldsymbol{\gamma}' = ST \left(I - \beta F T \right)^{-1} \tag{B.3}$$

B.2 The law of motion for X_t.

The law of motion is a slight generalisation of that presented in Woodford (2003*b*), extended here to allow the underlying state to have an arbitrary AR(1) coefficient (Woodford limited attention to the case of a random walk). With a state-space representation and Gaussian shocks, the optimal estimator is a Kalman filter. Agent j's q^{th}-order expectation (regarding $x_{t|t}^{(q-1)}$) is then given by:

$$E_t(j) \left[x_{t|t}^{(q-1)} \right] = E_{t-1}(j) \left[x_{t|t}^{(q-1)} \right] + k_q \left\{ s_t(j) - E_{t-1}(j) [s_t(j)] \right\} \tag{B.4}$$

where k_q is the time-invariant Kalman gain for the qth-order expectation, which is common to all agents as their information problems are symmetric. I start with the first-order expectation (about the underlying state variable itself):

$$E_t(i)[x_t] = E_{t-1}(i)[x_t] + k_1 \left\{ s_t(i) - E_{t-1}(i)[s_t(i)] \right\} \tag{B.5a}$$

where k_1 is the Kalman gain.[37] Substituting in the signal and law of motion for x_t, and recognising that u_t and $v_t(j)$ are not forecastable, this rearranges to

$$E_t(i)[x_t] = \rho k_1 x_{t-1} + \rho(1-k_1) E_{t-1}(i)[x_{t-1}] + k_1(u_t + v_t(i)) \tag{B.5b}$$

Taking the average across all agents then produces

$$x^{(1)}_{t|t} = \rho k_1 x_{t-1} + \rho(1-k_1) x^{(1)}_{t-1|t-1} + k_1 u_t \tag{B.5c}$$

where the law of large numbers allows me to drop the term in the average idiosyncratic shock. Equation (B.5c) is the law of motion for agents' first-order average expectation.

Second-order expectations are those about the average first-order expectation:

$$E_t(i)\left[x^{(1)}_{t|t}\right] = E_{t-1}(i)\left[x^{(1)}_{t|t}\right] + k_2\left\{s_t(i) - E_{t-1}(i)[s_t(i)]\right\} \tag{B.6a}$$

Substituting in (B.5c), this then becomes

$$\begin{aligned} E_t(i)\left[x^{(1)}_{t|t}\right] &= E_{t-1}(i)\left[\rho k_1 x_{t-1} + \rho(1-k_1) x^{(1)}_{t-1|t-1} + k_1 u_t\right] \\ &\quad + k_2\left\{\rho x_{t-1} + u_t + v_t(i) - \rho E_{t-1}(i)[x_{t-1}]\right\} \end{aligned} \tag{B.6b}$$

Rearranging and taking the average, together with the law of large numbers again for the idiosyncratic shocks, then produces the law of motion for agents' average second-order expectation:

$$x^{(2)}_{t|t} = \rho k_2 x_{t-1} + \rho(k_1 - k_2) x^{(1)}_{t-1|t-1} + \rho(1-k_2) x^{(2)}_{t-1|t-1} + k_2 u_t \tag{B.6c}$$

Continuing this process, the law of motion for the complete hierarchy of expectations follows a vector AR(1) process, with the transition matrix lower triangular:

$$X_t = FX_{t-1} + Gu_t \tag{B.7a}$$

$$F = \rho \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ k_1 & (1-k_1) & 0 & 0 & \cdots \\ k_2 & (k_1-k_2) & (1-k_1) & 0 & \cdots \\ k_3 & (k_2-k_3) & (k_1-k_2) & (1-k_1) & \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \qquad G = \begin{bmatrix} 1 \\ k_1 \\ k_2 \\ k_3 \\ \vdots \end{bmatrix} \tag{B.7b}$$

The optimal kalman filters

Let agent j's forecast errors about variable θ_t be denoted $\theta^{\text{err}}_{t|t-1}(i) \equiv \theta_t - E_{t-1}(i)[\theta_t]$. The sequence of optimal kalman gains are given by:

$$k_1 = Cov\left(x^{(0)}_{t|t}, s^{\text{err}}_{t|t-1}(i)\right)\left[Var\left(s^{\text{err}}_{t|t-1}(i)\right)\right]^{-1} \tag{B.8a}$$

$$k_2 = Cov\left(x^{(1)}_{t|t}, s^{\text{err}}_{t|t-1}(i)\right)\left[Var\left(s^{\text{err}}_{t|t-1}(i)\right)\right]^{-1} \tag{B.8b}$$

$$\vdots$$

[37] Strictly, i's period-t Kalman gain should be written as $k_{1,t}(i)$. However, since all agents problems are symmetric it will be common to all, and since x_t is stationary it will converge to a time invariant form, so that $k_{1,t}(i) = k_{1,t} \to k_1$.

Note that $s^{\text{err}}_{t|t-1}(i) = x^{(0):\text{err}}_{t|t-1}(i) + v_t(i)$ and that $x^{(q)}_{t|t}$ may be rewritten as $x^{(q)}_t = x^{(q):\text{err}}_{t|t-1}(i) + E_{t-1}(i)\left[x^{(q)}_t\right]$. Since the signal innovation is, by construction, orthogonal to all past information, it must be the case that $Cov\left(E_{t-1}(i)\left[x^{(q)}_t\right], s^{\text{err}}_{t|t-1}(i)\right) = 0$, so the kalman gains are given by:

$$k_1 = Cov\left(x^{(0):\text{err}}_{t|t-1}(i), x^{(0):\text{err}}_{t|t-1}(i) + v_t(i)\right)\left[Var\left(x^{(0):\text{err}}_{t|t-1}(i) + v_t(i)\right)\right]^{-1} \tag{B.9a}$$

$$k_2 = Cov\left(x^{(1):\text{err}}_{t|t-1}(i), x^{(0):\text{err}}_{t|t-1}(i) + v_t(i)\right)\left[Var\left(x^{(0):\text{err}}_{t|t-1}(i) + v_t(i)\right)\right]^{-1} \tag{B.9b}$$

$$\vdots$$

Let $Q = Cov\left(X^{\text{err}}_{t|t-1}(i)\right)$ be the *prior* error variance and $V = Cov\left(X^{\text{err}}_{t|t}(i)\right)$ be the *posterior* error variance, where $\{q_{jk}\}$ and $\{\nu_{jk}\}$ start at $j, k = 0$ for errors about the underlying state:

$$q_{jk} \equiv Cov\left(x^{(j):\text{err}}_{t|t-1}(i), x^{(k):\text{err}}_{t|t-1}(i)\right) \tag{B.10a}$$

$$\nu_{jk} \equiv Cov\left(x^{(j):\text{err}}_{t|t}(i), x^{(k):\text{err}}_{t|t}(i)\right) \tag{B.10b}$$

The kalman gains can then be written simply as

$$k_1 = q_{00}\left(q_{00} + \sigma_v^2\right)^{-1} \tag{B.11a}$$

$$k_2 = q_{10}\left(q_{00} + \sigma_v^2\right)^{-1} \tag{B.11b}$$

$$k_3 = q_{20}\left(q_{00} + \sigma_v^2\right)^{-1} \tag{B.11c}$$

$$\vdots$$

From the law of motion, the prior expectation errors follow

$$x^{(0):\text{err}}_{t|t-1}(i) = \rho\left\{x^{(0):\text{err}}_{t-1|t-1}(i)\right\} + u_t \tag{B.12a}$$

$$x^{(1):\text{err}}_{t|t-1}(i) = \rho\left\{k_1 x^{(0):\text{err}}_{t-1|t-1}(i) + (1-k_1)\, x^{(1):\text{err}}_{t-1|t-1}(i)\right\} + k_1 u_t \tag{B.12b}$$

$$x^{(2):\text{err}}_{t|t-1}(i) = \rho\left\{k_2 x^{(0):\text{err}}_{t-1|t-1}(i) + (k_1-k_2)\, x^{(1):\text{err}}_{t-1|t-1}(i) + (1-k_1)\, x^{(2):\text{err}}_{t-1|t-1}(i)\right\} + k_2 u_t \tag{B.12c}$$

$$\vdots$$

which implies that, for the convergent solution,

$$q_{00} = \rho^2\{\nu_{00}\} + \sigma_u^2 \tag{B.13a}$$

$$q_{10} = \rho^2\{k_1\nu_{00} + (1-k_1)\,\nu_{10}\} + k_1\sigma_u^2 \tag{B.13b}$$

$$q_{20} = \rho^2\{k_2\nu_{00} + (k_1-k_2)\,\nu_{10} + (1-k_1)\,\nu_{20}\} + k_2\sigma_u^2 \tag{B.13c}$$

$$\vdots$$

Similarly, from the definition of the kalman filter, the posterior expectation errors are:

$$x_{t|t}^{(0):\text{err}}(i) = (1-k_1)\, x_{t|t-1}^{(0):\text{err}}(i) - k_1 v_t(i) \tag{B.14a}$$
$$x_{t|t}^{(1):\text{err}}(i) = (1-k_2)\, x_{t|t-1}^{(1):\text{err}}(i) - k_2 v_t(i) \tag{B.14b}$$
$$x_{t|t}^{(2):\text{err}}(i) = (1-k_3)\, x_{t|t-1}^{(2):\text{err}}(i) - k_3 v_t(i) \tag{B.14c}$$
$$\vdots$$

which implies

$$\nu_{00} = (1-k_1)(1-k_1)\, q_{00} + k_1 k_1 \sigma_v^2 \tag{B.15a}$$
$$\nu_{10} = (1-k_2)(1-k_1)\, q_{10} + k_2 k_1 \sigma_v^2 \tag{B.15b}$$
$$\nu_{20} = (1-k_3)(1-k_1)\, q_{20} + k_3 k_1 \sigma_v^2 \tag{B.15c}$$
$$\vdots$$

The three systems of equations (B.11), (B.13) and (B.15) may then be solved recursively. Starting with equations (B.11a), (B.13a) and (B.15a), it is easy to see that

$$q_{00} = \rho^2 \left\{ \left(\frac{\sigma_v^2}{q_{00} + \sigma_v^2} \right)^2 q_{00} + \left(\frac{q_{00}}{q_{00} + \sigma_v^2} \right)^2 \sigma_v^2 \right\} + \sigma_u^2 \tag{B.16}$$

which solves as:[38]

$$q_{00} = \frac{\sigma_u^2}{2} \left\{ 1 - \left(1-\rho^2\right)\delta + \left(\left(1-\rho^2\right)^2 \delta^2 + 2\left(1+\rho^2\right)\delta + 1 \right)^{\frac{1}{2}} \right\} \tag{B.17}$$

where $\delta \equiv \frac{\sigma_v^2}{\sigma_u^2}$. The convergent value for k_1 is therefore

$$k_1 = \frac{1 - (1-\rho^2)\,\delta + \left((1-\rho^2)^2 \delta^2 + 2(1+\rho^2)\,\delta + 1 \right)^{\frac{1}{2}}}{1 - (1-\rho^2)\,\delta + \left((1-\rho^2)^2 \delta^2 + 2(1+\rho^2)\,\delta + 1 \right)^{\frac{1}{2}} + 2\delta} \tag{B.18}$$

from which it is clear that $k_1 \in (0,1)$ for positive δ and positive ρ, and that:

- $k_1 \to 1$ as $\delta \to 0$
- $k_1 \to 0$ as $\delta \to \infty$
- $k_1 \to 1/(1+\delta)$ as $\rho \to 0$
- $k_1 \to 1$ as $\rho \to \infty$

For subsequent elements, note that equation (B.13) may be rewritten as

$$q_{00} = \rho^2 \{\nu_{00}\} + \sigma_u^2 \tag{B.19a}$$
$$q_{j0} = k_j \sigma_u^2 + \rho^2 \left\{ k_j \nu_{00} + \sum_{l=1}^{j-1} (k_{j-l} - k_{j-l+1})\, \nu_{l0} \right\} + \rho^2 (1-k_1)\, \nu_{j0} \quad j \geq 1 \tag{B.19b}$$

[38] As a variance, q_{00} must be positive so the negative root may be ignored.

and that after partially substituting in (B.11), equation (B.15) may be rewritten as

$$\nu_{00} = (1-k_1)(1-k_1)\,q_{00} + k_1 k_1 \sigma_v^2 \tag{B.20a}$$

$$\nu_{j0} = \left(1 - \frac{q_{j0}}{q_{00}+\sigma_v^2}\right)(1-k_1)\,q_{j0} + \frac{q_{j0}}{q_{00}+\sigma_v^2} k_1 \sigma_v^2 \qquad j \geq 1 \tag{B.20b}$$

Substituting (B.20b) into (B.19b) then shows that q_{j0} must satisfy the quadratic:

$$a\,(q_{j0})^2 + b\,(q_{j0}) + c_j = 0 \tag{B.21a}$$

where

$$a = \rho^2 (1-k_1)^2 \left(\frac{1}{q_{00}+\sigma_v^2}\right) \tag{B.21b}$$

$$b = 1 - \rho^2 (1-k_1)^2 (1+k_1) \tag{B.21c}$$

$$c_j = -\left\{ k_j \sigma_u^2 + \rho^2 \left\{ k_j \nu_{00} + \sum_{l=1}^{j-1} (k_{j-l} - k_{j-l+1})\,\nu_{l0} \right\} \right\} \tag{B.21d}$$

Note that a and b are common for all j, and that c_j will be pre-determined when calculating q_{j0}, so solving (B.21a) gives q_{j0} exactly (i.e. there is no need to find a fixed point through iteration). Finally, also note that since agents' signals of the underlying shock are unbiased and agents are rational, then the covariances of all higher-order expectations with the underlying shock must be positive ($Cov\left(x_{t|t}^{(q)}, x_t\right) > 0\ \forall q$), the negative roots to the solutions of (B.21a) may be rejected.

B.3 Nesting of, and convergence towards, the full-information forward solution.

Under full information, it must be the case that $x_{t|t}^{(q)} = x_t\ \forall\, q$, so that $F = \rho \begin{bmatrix} \mathbf{1}_{\infty\times 1} & \mathbf{0}_{\infty\times\infty} \end{bmatrix}$ and $G = \begin{bmatrix} \mathbf{1}_{\infty\times 1} \end{bmatrix}$, from which it is clear that (B.2) collapses to (7a). That the solution under idiosyncratically noisy information converges to this smoothly as $\sigma_v^2 \to 0$ follows directly from the optimality of the Kalman filter.

C The NKPC with ICK and positive trend inflation

This appendix derives the New Keynesian Phillips Curve when firms face imperfect common knowledge. In contrast to Nimark (2008), I here include both (i) the possibility of non-zero trend inflation and (ii) ongoing uncertainty about past price levels. Much of the derivation extends the standard treatment of, for example, Ascari and Sbordone (2014) although the expressions for the present discount values of expected future marginal revenue and cost are written in nominal terms rather than real.

C.1 The (representative) household and central bank

The representative household and central bank are unchanged from the textbook model. I present usual the log-linearised equations without comment:

$$y_t = E_t^{\Omega}\left[y_{t+1}\right] - \sigma\left(i_t - E_t^{\Omega}\left[p_{t+1}\right] + p_t - x_t\right) \tag{C.1}$$

$$w_t - p_t = \frac{1}{\psi} n_t + \frac{1}{\sigma} y_t \tag{C.2}$$

$$i_t = \phi_\pi\left(p_t - p_{t-1}\right) + \phi_y y_t \tag{C.3}$$

C.2 Production, demand and pricing

Firm production is in terms of labour, with decreasing marginal productivity:

$$Y_{i,t} = N_{i,t}^{1-\alpha} \tag{C.4}$$

Firm demand comes from a Dixit-Stiglitz aggregator:

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\varepsilon} Y_t \tag{C.5}$$

Firms face Calvo pricing frictions. Let the firm's (nominal) reset price be Q_{it} and the probability of not resetting each period be θ. The aggregate price level evolves as:

$$\begin{aligned} P_t^{1-\varepsilon} &= \int_0^1 P_{i,t}^{1-\varepsilon} di \\ &= \theta P_{t-1}^{1-\varepsilon} + (1-\theta)\int_0^1 Q_{i,t}^{1-\varepsilon} di \end{aligned} \tag{C.6}$$

When able to reset, firms seek to maximise their expected flow of real profits:

$$\max_{Q_{it}} E_{i,t}\left[\sum_{j=0}^{\infty} \frac{\Lambda_{t,j}}{\Lambda_{t,0}} (\beta\theta)^j \left\{\frac{Q_{it}}{P_t} Y_{i,t+j}|_{P_{i,t+j}=Q_{i,t}} - \frac{W_{t+j}}{P_{t+j}} N_{i,t+j}|_{P_{i,t+j}=Q_{i,t}}\right\}\right] \tag{C.7}$$

subject to (C.4) and (C.5), and where $\Lambda_{t,j} = C_{t+j}^{-\sigma} = Y_{t+j}^{-\sigma}$, such that $\beta^s \frac{\Lambda_{t,j}}{\Lambda_{t,0}}$ is the (real) stochastic discount factor for period $t+j$ from the perspective of period t. Substituting these both in, the problem is then:

$$\max_{Q_{it}} E_{i,t}\left[\sum_{j=0}^{\infty} \frac{\Lambda_{t,j}}{\Lambda_{t,0}} (\beta\theta)^j \left\{\left(\frac{Q_{i,t}}{P_{t+j}}\right)^{1-\varepsilon} Y_{t+j} - \left(\frac{W_{t+j}}{P_{t+j}}\right)(Y_{t+j})^{\frac{1}{1-\alpha}} \left(\frac{Q_{i,t}}{P_{t+j}}\right)^{\frac{-\varepsilon}{1-\alpha}}\right\}\right] \tag{C.8}$$

The first-order condition is:

$$\begin{aligned} &(1-\varepsilon) E_{i,t}\left[\sum_{j=0}^{\infty} \frac{\Lambda_{t,j}}{\Lambda_{t,0}} (\beta\theta)^j \left(\frac{Q_{i,t}}{P_{t+j}}\right)^{-\varepsilon} \frac{Y_{t+j}}{P_{t+j}}\right] \\ &= \left(\frac{-\varepsilon}{1-\alpha}\right) E_{i,t}\left[\sum_{j=0}^{\infty} \frac{\Lambda_{t,j}}{\Lambda_{t,0}} (\beta\theta)^j \left(\frac{W_{t+j}}{P_t}\right)(Y_{t+j})^{\frac{1}{1-\alpha}} \left(\frac{Q_{i,t}}{P_{t+j}}\right)^{\frac{-\varepsilon}{1-\alpha}-1} \frac{1}{P_{t+j}}\right] \end{aligned} \tag{C.9}$$

or, gathering the terms in Q_{it},

$$Q_{i,t}^{\left(1+\frac{\alpha\varepsilon}{1-\alpha}\right)} = \left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{1}{1-\alpha}\right)\frac{E_{i,t}\left[\Psi_t\right]}{E_{i,t}\left[\Phi_t\right]} \tag{C.10}$$

where

$$\Psi_t = \sum_{j=0}^{\infty}(\beta\theta)^j\left(\frac{Y_{t+j}}{Y_t}\right)^{-\sigma}\left(\frac{W_{t+j}}{P_{t+j}}\right)(Y_{t+j})^{\frac{1}{1-\alpha}}(P_{t+j})^{\frac{\varepsilon}{1-\alpha}} \tag{C.11}$$

$$\Phi_t = \sum_{j=0}^{\infty}(\beta\theta)^j\left(\frac{Y_{t+j}}{Y_t}\right)^{-\sigma}(P_{t+j})^{\varepsilon-1}Y_{t+j} \tag{C.12}$$

and I have substituted in for $\Lambda_{t,j}$. Note that Ψ_t and Φ_t can be written recursively:

$$\Psi_t = \left(\frac{W_t}{P_t}\right)Y_t^{\frac{1}{1-\alpha}}P_t^{\frac{\varepsilon}{1-\alpha}} + (\beta\theta)\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\Psi_{t+1} \tag{C.13}$$

$$\Phi_t = P_t^{\varepsilon-1}Y_t + (\beta\theta)\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\Phi_{t+1} \tag{C.14}$$

Trend values of Ψ and Φ (note that nominal variables still get a time subscript when at trend if $\overline{\Pi} \neq 1$):

$$\overline{\Phi}_t = \left(\frac{\overline{Y}}{1-\beta\theta\overline{\Pi}^{\varepsilon-1}}\right)\overline{P}_t^{\varepsilon-1} \tag{C.15}$$

$$\overline{\Psi}_t = \left(\frac{\overline{\left(\frac{W}{P}\right)}\overline{Y}^{\frac{1}{1-\alpha}}}{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}}\right)\overline{P}_t^{\frac{\varepsilon}{1-\alpha}} \tag{C.16}$$

Linearising everything:

$$p_t = \theta\overline{\Pi}^{\varepsilon-1}p_{t-1} + \left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)\int_0^1 q_{i,t}\,di \tag{C.17}$$

$$q_{i,t} = \left(\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}\right)E_{i,t}\left[\psi_t - \phi_t\right] \tag{C.18}$$

$$\begin{aligned}\psi_t = {} & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(w_t - p_t + \left(\frac{1}{1-\alpha}\right)y_t + \frac{\varepsilon}{1-\alpha}p_t\right) \\ & + \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\left(\psi_{i,t+1} - \sigma\left(y_{t+1}-y_t\right)\right)\end{aligned} \tag{C.19}$$

$$\phi_t = \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}\right)\left(y_t + (\varepsilon-1)\,p_t\right) + \beta\theta\overline{\Pi}^{\varepsilon-1}\left(\phi_{t+1} - \sigma\left(y_{t+1}-y_t\right)\right) \tag{C.20}$$

C.3 Aggregation and price dispersion

Labour market clearing implies:

$$\begin{aligned}N_t &= \int_0^1 N_{i,t}\,di \\ &= \int_0^1 (Y_{i,t})^{\frac{1}{1-\alpha}}\,di \\ &= (Y_t)^{\frac{1}{1-\alpha}}\underbrace{\int_0^1\left(\frac{P_{i,t}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}}di}_{S_t}\end{aligned} \tag{C.21}$$

With Calvo pricing, S_t must follow:

$$S_t = (1-\theta)\int_0^1 \left(\frac{Q_{i,t}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di + \theta \Pi_t^{\frac{\varepsilon}{1-\alpha}} S_{t-1} \tag{C.22}$$

When at trend, this is:

$$\overline{S} = \frac{(1-\theta)\overline{\left(\frac{Q}{P}\right)}^{\frac{-\varepsilon}{1-\alpha}}}{1-\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}} \tag{C.23}$$

Log-linearising around trend then gives:

$$n_t = \left(\frac{1}{1-\alpha}\right) y_t + s_t \tag{C.24}$$

$$s_t = \left(1-\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(\frac{-\varepsilon}{1-\alpha}\right)\left(\int_0^1 q_{i,t} di - p_t\right) + \theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\left(s_{t-1} + \frac{\varepsilon}{1-\alpha}\left(p_t - p_{t-1}\right)\right) \tag{C.25}$$

or, substituting in equation (C.17) to replace the average reset price,

$$s_t = \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right) s_{t-1} + \left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\left(p_t - p_{t-1}\right) \tag{C.26}$$

C.4 The full system

Bringing everything together, I have:

$$y_t = E_t^{\Omega}\left[y_{t+1}\right] - \sigma\left(i_t - E_t^{\Omega}\left[p_{t+1}\right] + p_t - x_t\right) \tag{C.27}$$

$$i_t = \phi_\pi\left(p_t - p_{t-1}\right) + \phi_y y_t \tag{C.28}$$

$$p_t = \theta\overline{\Pi}^{\varepsilon-1} p_{t-1} + \left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)\left(\frac{1-\alpha}{1-\alpha+\alpha\varepsilon}\right)\overline{E}_t\left[\psi_t - \phi_t\right] \tag{C.29}$$

$$\psi_t = \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left((1+\omega)\, y_t + \frac{1}{\psi} s_t + \frac{\varepsilon}{1-\alpha} p_t\right) + \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\left(\psi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{C.30}$$

$$\phi_t = \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}\right)\left(y_t + (\varepsilon-1)\, p_t\right) + \beta\theta\overline{\Pi}^{\varepsilon-1}\left(\phi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{C.31}$$

$$s_t = \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right) s_{t-1} + \left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\left(p_t - p_{t-1}\right) \tag{C.32}$$

where $\omega = \frac{1}{\sigma} - 1 + \left(\frac{1}{1-\alpha}\right)\left(1 + \frac{1}{\psi}\right)$. The final four equations together represent the Phillips curve.

C.5 Linear production and zero trend inflation

With constant returns to scale in production ($\alpha = 0$) and zero trend inflation ($\overline{\Pi} = 1$), equations (C.29)-(C.32) become:

$$p_t = \theta p_{t-1} + (1-\theta)\,\overline{E}_t\left[\psi_t - \phi_t\right] \tag{C.33}$$

$$\psi_t = (1-\beta\theta)\left((1+\omega)\, y_t + \varepsilon\, p_t\right) + \beta\theta\left(\psi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{C.34}$$

$$\phi_t = (1-\beta\theta)\left(y_t + (\varepsilon-1)\, p_t\right) + \beta\theta\left(\phi_{t+1} - \sigma\left(y_{t+1} - y_t\right)\right) \tag{C.35}$$

$$s_t = 0 \tag{C.36}$$

In this case, I can define $q_t^* \equiv \psi_t - \phi_t$ and write:

$$p_t = \theta p_{t-1} + (1-\theta)\,\overline{E}_t\left[q_t^*\right] \tag{C.37}$$

$$q_t^* = (1-\beta\theta)\,(\omega y_t + p_t) + \beta\theta q_{t+1}^* \tag{C.38}$$

where ωy_t is the average real marginal cost and q_t^* is the optimal reset price. To recover the expressions in the main text, note that:

$$\begin{aligned}
\overline{E}_t\left[q_{t+1}^*\right] &= \int_0^1 E_t(i)\left[q_{t+1}^*\right] di \\
&= \int_0^1 E_t(i)\left[E_{t+1}(i)\left[q_{t+1}^*\right]\right] di \\
&= \int_0^1 E_t(i)[q_{t+1}(i)]\, di \\
&= \overline{E}_t\left[q_{t+1}\right] \\
&= \overline{E}_t\left[\frac{p_{t+1} - \theta p_t}{1-\theta}\right]
\end{aligned} \tag{C.39}$$

where the second equality uses the fact that the law of iterated expectations must apply to individual information sets when they are only increasing over time; the fourth equality uses the symmetry of firms' problems; and the final equality substitutes in from $p_t = \theta p_{t-1} + (1-\theta)\,q_t$. Substituting this back in then gives equation (36) in the main text.

D Proof of proposition 4

The solution coefficients γ and λ may readily be calculated numerically with any of a variety of standard solution methods. To derive algebraic expressions for them, I first combine the CB's decision rule, the HH's Euler equation and the NKPC to produce a single competitive equilibrium condition.

D.1 Obtaining a single competitive equilibrium condition

Combining the central bank's decision rule with the household's Euler equation and substituting forward, I obtain:[39]

$$\begin{aligned}
y_t = \sigma\delta\,(1-\delta\rho)^{-1} \quad & x_t \\
+\,\sigma\delta\,\phi_\pi \quad & p_{t-1} \\
-\,\sigma\delta\,(1-\phi_\pi\delta+\phi_\pi) \quad & p_t \\
+\,\sigma\delta\,(1-\delta\phi_\pi)\,(1-\delta)\,\sum_{s=0}^{\infty}\delta^s E_t^{\Omega}\left[p_{t+s+1}\right] \quad &
\end{aligned} \tag{D.1}$$

[39] A limiting term of $\lim_{s\to\infty}\delta^s E_t^{\Omega}\left[y_{t+s+1}\right]$ has been implicitly set to zero in (D.1). Since transversality is satisfied *by definition* in purely forward-looking solutions and I later demonstrate the inadmissibility of backward-looking solutions, its absence here is innocuous.

Substituting (D.1) into (36) then gives the model's equilibrium condition:

$$\begin{aligned} p_t = b_p \overline{E}_t\left[x_t\right] + \theta p_{t-1} &+ \zeta_{-1}\overline{E}_t\left[p_{t-1}\right] \\ &+ \zeta_0 \ \overline{E}_t\left[p_t\right] \\ &+ \beta\theta \ \overline{E}_t\left[p_{t+1}\right] \\ &+ \zeta_{1^+}\overline{E}_t\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s p_{t+s+1}\right] \end{aligned} \tag{D.2a}$$

This gives the current log deviation of the price level from its steady-state path in terms of (i) the previous period's log deviation; (ii) firms' average expectation of the current value of the underlying shock process; and (iii) firms' average expectations of the past, current and all future price levels (note that p_{t+1} appears in both of the bottom two lines). The compound parameters are given by:

$$b_p = \theta\kappa\sigma\delta\left(1-\delta\rho\right)^{-1} \tag{D.2b}$$

$$\zeta_{-1} = \theta\kappa\sigma\delta\phi_\pi \tag{D.2c}$$

$$\zeta_0 = 1-\theta\left(1+\beta\right)-\theta\kappa\sigma\delta\left(1-\phi_\pi\delta+\phi_\pi\right) \tag{D.2d}$$

$$\zeta_{1^+} = \theta\kappa\sigma\delta\left(1-\phi_\pi\delta\right) \tag{D.2e}$$

Although perhaps unusual, (D.2) is a perfectly valid statement of the equilibrium condition underlying Galí (2008), extended here only to accomodate incomplete information among price-setting firms. Note that the term on the final line of (D.2a) is a weighted average of all future price deviations. When $\phi_y > 0$ it is skewed in favour of the near-term, while when $\phi_y = 0$ it is a simple average. Regardless, since trend inflation is assumed to be zero, it follows that $\lim_{\phi_y\to 0}(1-\delta)\sum_{s=0}^{\infty}\delta^s p_{t+s+1} = \lim_{s\to\infty} p_{t+s}$. This will be non-zero for any $x_t \neq 0$ if prices exhibit a unit root, as in the standard solution to the NK model.

D.2 Solving the model under full information

Under full information the competitive equilibrium condition (D.2) simplifies to:

$$p_t = \left(\frac{1}{1-\zeta_0}\right)\left(b_p x_t + \left(\theta+\zeta_{-1}\right)p_{t-1} + \beta\theta E_t^{\Omega}\left[p_{t+1}\right] + \zeta_{1^+}E_t^{\Omega}\left[(1-\delta)\sum_{s=0}^{\infty}\delta^s p_{t+s+1}\right]\right) \tag{D.3}$$

While firms' expectation of future prices must be formed as:

$$E_t^{\Omega}\left[p_{t+1}\right] = \gamma\left(\rho+\lambda\right)x_t + \lambda^2 p_{t-1} \tag{D.4a}$$

$$E_t^{\Omega}\left[p_{t+2}\right] = \gamma\left(\rho^2+\lambda\rho+\lambda^2\right)x_t + \lambda^3 p_{t-1} \tag{D.4b}$$

$$E_t^{\Omega}\left[p_{t+3}\right] = \gamma\left(\rho^3+\lambda\rho^2+\lambda^2\rho+\lambda^3\right)x_t + \lambda^4 p_{t-1} \tag{D.4c}$$

$$\vdots$$

Substituting (D.4) into (D.3) then gives

$$p_t = \left(\frac{1}{1-\zeta_0}\right)\left(\begin{array}{l} \quad (b_p + \beta\theta\gamma(\rho+\lambda))\, x_t \\ + (\theta + \zeta_{-1} + \beta\theta\lambda^2)\, p_{t-1} \\ + \zeta_{1^+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\left(\gamma\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right)x_t + \lambda^{s+2}p_{t-1}\right) \end{array}\right) \tag{D.5}$$

Gathering like terms, it follows that

$$\gamma = \left(\frac{1}{1-\zeta_0}\right)\left(b_p + \beta\theta\gamma(\rho+\lambda) + \gamma\,\zeta_{1^+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right)\right) \tag{D.6a}$$

$$\lambda = \left(\frac{1}{1-\zeta_0}\right)\left(\theta + \zeta_{-1} + \beta\theta\lambda^2 + \lambda^2\zeta_{1^+}\left(\frac{1-\delta}{1-\delta\lambda}\right)\right) \tag{D.6b}$$

The coefficient γ

Starting with the expression for γ, note that (D.6a) may be rewritten as

$$\gamma = \frac{b_p}{\xi} \quad \text{where} \quad \xi = 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\left(\sum_{q=0}^{s+1}\rho^q\lambda^{s+1-q}\right) \tag{D.7a}$$

The expression for ξ can then be re-expressed as:

$$\begin{aligned} \xi &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\lambda^{s+1}\left(\sum_{q=0}^{s+1}\left(\frac{\rho}{\lambda}\right)^q\right) \\ &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\sum_{s=0}^{\infty}\delta^s\lambda^{s+1}\left(\frac{1-\left(\frac{\rho}{d}\right)^{s+2}}{1-\frac{\rho}{\lambda}}\right) \\ &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\left(\frac{\lambda}{1-\frac{\rho}{\lambda}}\right)\sum_{s=0}^{\infty}(\delta\lambda)^s\left(1-\left(\frac{\rho}{\lambda}\right)^{s+2}\right) \\ &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\left(\frac{\lambda}{1-\frac{\rho}{\lambda}}\right)\left(\frac{1}{1-\delta\lambda} - \left(\frac{\rho}{\lambda}\right)^2\frac{1}{1-\delta\rho}\right) \end{aligned} \tag{D.8}$$

where the final equality requires that $\delta\lambda < 1$. For values of $\lambda \geq \frac{1}{\delta}$, the sum $\sum_{s=0}^{\infty}(\delta\lambda)^s$ will explode, leading to $\gamma = 0$ (that is, non-existence of a solution).[40] The expression (D.8) simplifies further as

$$\begin{aligned} \xi &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\left(\frac{1}{\lambda-\rho}\right)\left(\frac{\lambda^2}{1-\delta\lambda} - \frac{\rho^2}{1-\delta\rho}\right) \\ &= 1 - \zeta_0 - \beta\theta(\rho+\lambda) - \zeta_{1^+}(1-\delta)\left(\frac{\lambda+\rho-\delta\rho\lambda}{(1-\delta\lambda)(1-\delta\rho)}\right) \end{aligned} \tag{D.9}$$

[40] Note that since $\rho \in (0,1)$ and $\delta \in (0,1]$, it must be the case that $\delta\rho < 1$. Also note that the third equality does not require that $\frac{\rho}{\lambda} < 1$ in order to write $\left(\frac{1-\left(\frac{\rho}{\lambda}\right)^{s+2}}{1-\frac{\rho}{\lambda}}\right)$, as the latter is simplifying a finite (rather than infinite) sum.

Expanding ζ_0 and ζ_{1^+}, this then becomes

$$\begin{aligned}\xi &= \theta(1+\beta) + \theta\kappa\sigma\delta(\phi_\pi + 1 - \phi_\pi\delta) - \beta\theta(\rho+\lambda) \\ &\quad - \theta\kappa\sigma\delta(1-\phi_\pi\delta)(1-\delta)\left(\frac{\lambda+\rho-\delta\rho\lambda}{(1-\delta\lambda)(1-\delta\rho)}\right)\end{aligned} \tag{D.10}$$

or, after some straightforward manipulation,

$$\xi = \theta + \beta\theta(1-\rho-\lambda) + \theta\kappa\sigma\left(1 - \frac{(1-\delta)}{(1-\delta\lambda)}\frac{(1-\delta\phi_\pi)}{(1-\delta\rho)}\right) \tag{D.11}$$

The coefficient λ

Next looking at the expression for λ, we can rewrite (D.6b) as

$$\left\{\beta\theta\delta\right\}\lambda^3 - \left\{\beta\theta + (1-\zeta_0)\delta + \zeta_{2^+}\right\}\lambda^2 + \left\{1 - \zeta_0 + (\theta+\zeta_{-1})\delta\right\}\lambda - \left\{\theta + \zeta_{-1}\right\} = 0 \tag{D.12}$$

Expanding the latter three compound parameters, we have

$$\left\{\beta\theta + (1-\zeta_0)\delta + \zeta_{2^+}\right\} = \beta\theta + \delta\theta(1+\beta) + \theta\kappa\sigma\delta \tag{D.13a}$$

$$\left\{1 - \zeta_0 + (\theta+\zeta_{-1})\delta\right\} = \beta\theta + \theta(1+\delta) + \theta\kappa\sigma\delta(1+\phi_\pi) \tag{D.13b}$$

$$\left\{\theta + \zeta_{-1}\right\} = \theta + \theta\kappa\sigma\delta\phi_\pi \tag{D.13c}$$

It is easy to confirm that (D.12) has a root of $\lambda = 1$:

$$\left\{\beta\theta\delta\right\}(1)^3 - \left\{\beta\theta + (1-\zeta_0)\delta + \zeta_{2^+}\right\}(1)^2 + \left\{1 - \zeta_0 + (\theta+\zeta_{-1})\delta\right\}(1) - \left\{\theta + \zeta_{-1}\right\} = 0 \tag{D.14}$$

Given this, (D.12) may be rewritten as:

$$(\lambda - 1)\left(\left\{\beta\theta\delta\right\}\lambda^2 - \left\{\beta\theta + \delta\theta + \theta\kappa\sigma\delta\right\}\lambda + \left\{\theta + \theta\kappa\sigma\delta\phi_\pi\right\}\right) = 0 \tag{D.15}$$

from which the other two roots may be readily obtained as

$$\lambda = \frac{\beta+\delta+\kappa\sigma\delta}{2\beta\delta} \pm \frac{\sqrt{(\beta+\delta+\kappa\sigma\delta)^2 - 4\beta\delta(1+\kappa\sigma\delta\phi_\pi)}}{2\beta\delta} \tag{D.16}$$

These are the non-zero eigenvalues of the system highlighted in the main text. To see that the solution is the lower envelope of these, start from equation (40) in the main text. Cho and Moreno (2011) show that substituting this expression forward gives:

$$\boldsymbol{\zeta}_t = M_k E_t^\Omega[\boldsymbol{\zeta}_{t+k}] + \Lambda_k\boldsymbol{\zeta}_{t-1} + \Gamma_k x_t \tag{D.17a}$$

where $M_1 = A$, $\Lambda_1 = B$, $\Gamma_1 = C$ and, for $k \geq 2$,

$$M_k = (I - A\Lambda_{k-1})^{-1} A M_{k-1} \tag{D.17b}$$

$$\Lambda_k = (I - A\Lambda_{k-1})^{-1} B \tag{D.17c}$$

$$\Gamma_k = (I - A\Lambda_{k-1})^{-1}(C + A\Gamma_{k-1}\rho) \tag{D.17d}$$

so that, in the limit,

$$\boldsymbol{\zeta}_t = \Lambda \boldsymbol{\zeta}_{t-1} + \Gamma x_t + \lim_{k\to\infty} M_k E_t^{\Omega}\left[\boldsymbol{\zeta}_{t+k}\right] \tag{D.18}$$

where $\Lambda = \lim_{k\to\infty} \Lambda_k$ and $\Gamma = \lim_{k\to\infty} \Gamma_k$ and under the purely forward-looking solution the limiting expectation term (which accomodates backward-looking solutions) is zero. Since the eigenvalues of D are all distinct, the model must have a dominant solvent (S_1) and a minimal solvent (S_2), where

$$\min\left\{|\lambda| : \lambda \in \lambda\left(S_1\right)\right\} > \max\left\{|\lambda| : \lambda \in \lambda\left(S_2\right)\right\} \tag{D.19}$$

When S_1 and S_2 exist (as they do here), Rendahl (2017) proves that the sequence (D.17c) must converge to S_2, provided that $\Lambda_1 \neq S_1$. But since we have $\Lambda_1 = B$, the proof is established. Given the simplicity of the basic NK model, it is also straightforward here to confirm convergence to the minimal solution numerically.

E Proof of proposition 5

Recall that the candidate solution is of the form:

$$Z_t \equiv \begin{bmatrix} x_t & p_{t-1} & \widetilde{x}_{t|t} & \widetilde{p}_{t-1|t} \end{bmatrix}' \tag{E.1a}$$

$$Z_t = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \theta & \alpha_3 & \alpha_4 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{A} Z_{t-1} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ b_3 \\ b_4 \end{bmatrix}}_{B} u_t \tag{E.1b}$$

$$p_t = \underbrace{\begin{bmatrix} 0 & \theta & \alpha_3 & \alpha_4 \end{bmatrix}}_{\boldsymbol{\alpha}'} Z_t \tag{E.1c}$$

where I have filled in some elements of A, B and $\boldsymbol{\alpha}$ directly from the given law of motion for x_t and the equilibrium condition. Given this solution, firms' signal vectors are expressible as:

$$\boldsymbol{s}_t\left(i\right) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{N} Z_t + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{O} \boldsymbol{v}_t\left(i\right) \tag{E.2}$$

E.1 Firms' expectations

Without full information, individual firms must form expectations about the current state of the economy (Z_t). Since firms' signals may be written as $\boldsymbol{s}_t\left(j\right) = NZ_t + \sigma_v^2 I_2$, the model is in state-space form and the Bayes-rational estimator is the Kalman filter:

$$E_t(j)[Z_t] = E_{t-1}(j)[Z_t] + M_t\left\{\boldsymbol{s}_t\left(j\right) - E_{t-1}(j)[\boldsymbol{s}_t\left(j\right)]\right\} \tag{E.3}$$

where M_t is the (4×2) Kalman gain, common to all firms as their problems are symmetric. Defining $V_{t|t-1} \equiv Var\left(Z_t - E_{t-1}(j)[Z_t]\right)$ as the variance of firms' prior expectation errors, then for a given law of motion, the optimal filter converges to a time-invariant $M \equiv \begin{bmatrix} \boldsymbol{m}_x & \boldsymbol{m}_p \end{bmatrix}$ that satisfies:[41]

$$M = VN'\left(NVN' + \sigma_v^2 I_2\right)^{-1} \tag{E.4a}$$

$$V = A\left(V - VN'\left(NVN' + \sigma_v^2 I_2\right)^{-1} NV\right)A' + \sigma_u^2 BB' \tag{E.4b}$$

E.2 Reduced-form coefficients

Writing the solution as $p_t = \boldsymbol{\alpha}_p' Z_t$, simple inspection of the equilibrium condition (D.2) is sufficient to note that $\boldsymbol{\alpha}_p' = \begin{bmatrix} 0 & \theta & \alpha_3 & \alpha_4 \end{bmatrix}$. Next, see that it must be the case that (i) under full information, $\widetilde{x}_{t|t} = x_t$ and $\widetilde{p}_{t-1|t} = p_{t-1}$; and (ii) $\widetilde{x}_{t|t} \to x_t$ and $\widetilde{p}_{t-1|t} \to p_{t-1}$ as $\sigma_v^2 \to 0$ by the optimality of the Kalman filter. It therefore follows that $\boldsymbol{\alpha}_p$ must be consistent with the solution under full information (43), so that:

$$\boldsymbol{\alpha}_p' = \begin{bmatrix} 0 & \theta & \gamma & \lambda - \theta \end{bmatrix} \tag{E.5}$$

E.3 Determining the law of motion

The law of motion for x_t is given and the law of motion for p_{t-1} comes from the solution for $\boldsymbol{\alpha}$ shown above, so I here focus on those for $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$. The process for deriving them is equivalent to that in Woodford (2003*b*). First, note that given their definitions, we can write:

$$\widetilde{x}_{t|t} = \underbrace{\begin{bmatrix} (1-\varphi) & 0 & \varphi & 0 \end{bmatrix}}_{\boldsymbol{\varphi}_x'} \overline{E}_t\left[Z_t\right] \tag{E.6a}$$

$$\widetilde{p}_{t-1|t} = \underbrace{\begin{bmatrix} 0 & (1-\varphi) & 0 & \varphi \end{bmatrix}}_{\boldsymbol{\varphi}_p'} \overline{E}_t\left[Z_t\right] \tag{E.6b}$$

or, rearranging these,

$$\overline{E}_t\left[\widetilde{x}_{t|t}\right] = \frac{1}{\varphi}\left(\widetilde{x}_{t|t} - (1-\varphi)\,\overline{E}_t\left[x_t\right]\right) \tag{E.7a}$$

$$\overline{E}_t\left[\widetilde{p}_{t-1|t}\right] = \frac{1}{\varphi}\left(\widetilde{p}_{t-1|t} - (1-\varphi)\,\overline{E}_t\left[p_{t-1}\right]\right) \tag{E.7b}$$

Next, write agents' Kalman filter for Z_t:

$$E_t(i)[Z_t] = E_{t-1}(i)[Z_t] + M\left\{\boldsymbol{s}_t(i) - E_{t-1}(i)[\boldsymbol{s}_t(i)]\right\} \tag{E.8}$$

[41] For a derivation, see Hamilton (1994).

where $M = \begin{bmatrix} \boldsymbol{m}_x & \boldsymbol{m}_p \end{bmatrix}$ is a (4×2) Kalman gain matrix to be determined. Expanding this out and taking the average gives:

$$\overline{E}_t\left[Z_t\right] = A\overline{E}_{t-1}\left[Z_{t-1}\right] + M\left\{N\left(AZ_{t-1} + Bu_t\right) - NA\overline{E}_{t-1}\left[Z_{t-1}\right]\right\} \tag{E.9}$$

Gathering like terms and then substituting this into (E.6) then gives:

$$\widetilde{x}_{t|t} = \boldsymbol{\varphi}_x'\left(\left(I - MN\right)A\overline{E}_{t-1}\left[Z_{t-1}\right] + MNAZ_{t-1} + MNBu_t\right) \tag{E.10a}$$

$$\widetilde{p}_{t-1|t} = \boldsymbol{\varphi}_p'\left(\left(I - MN\right)A\overline{E}_{t-1}\left[Z_{t-1}\right] + MNAZ_{t-1} + MNBu_t\right) \tag{E.10b}$$

Note that NA and NB are given by:

$$NA = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \theta & \gamma & \lambda - \theta \end{bmatrix} \qquad NB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{E.11}$$

E.3.1 The law of motion for $\widetilde{x}_{t|t}$

Stepping (E.7) back one period, we can expand (E.10a) to read:

$$\begin{aligned} \widetilde{x}_{t|t} = {} & \left(\boldsymbol{\varphi}_x'A - \boldsymbol{\varphi}_x'MNA\right)\begin{bmatrix} \overline{E}_{t-1}\left[x_{t-1}\right] \\ \overline{E}_{t-1}\left[p_{t-2}\right] \\ \frac{1}{\varphi}\left(\widetilde{x}_{t-1|t-1} - \left(1-\varphi\right)\overline{E}_{t-1}\left[x_{t-1}\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{p}_{t-2|t-1} - \left(1-\varphi\right)\overline{E}_{t-1}\left[p_{t-2}\right]\right) \end{bmatrix} \\ & + \boldsymbol{\varphi}_x'MNAZ_{t-1} + \boldsymbol{\varphi}_x'MNBu_t \end{aligned} \tag{E.12}$$

Expanding $\boldsymbol{\varphi}_x'A$ and NA and NB, and then gathering like terms, this gives:

$$\begin{aligned} \widetilde{x}_{t|t} = {} & \left\{\boldsymbol{\varphi}_x'\boldsymbol{m}_x\rho\right\}x_{t-1} \\ & + \left\{\boldsymbol{\varphi}_x'\boldsymbol{m}_p\theta\right\}p_{t-2} \\ & + \left\{a_{33} - \boldsymbol{\varphi}_x'\boldsymbol{m}_p\frac{\gamma}{\varphi} + \boldsymbol{\varphi}_x'\boldsymbol{m}_p\gamma\right\}\widetilde{x}_{t-1|t-1} \\ & + \left\{a_{34} - \boldsymbol{\varphi}_x'\boldsymbol{m}_p\frac{\lambda-\theta}{\varphi} + \boldsymbol{\varphi}_x'\boldsymbol{m}_p\left(\lambda-\theta\right)\right\}\widetilde{p}_{t-2|t-1} \\ & + \left\{\left(1-\varphi\right)\rho + a_{31}\varphi - a_{33}\left(1-\varphi\right) - \boldsymbol{\varphi}_x'\boldsymbol{m}_x\rho + \boldsymbol{\varphi}_x'\boldsymbol{m}_p\frac{\gamma}{\varphi}\left(1-\varphi\right)\right\}\overline{E}_{t-1}\left[x_{t-1}\right] \\ & + \left\{a_{32}\varphi - a_{34}\left(1-\varphi\right) - \boldsymbol{\varphi}_x'\boldsymbol{m}_p\theta + \boldsymbol{\varphi}_x'\boldsymbol{m}_p\frac{\lambda-\theta}{\varphi}\left(1-\varphi\right)\right\}\overline{E}_{t-1}\left[p_{t-2}\right] \\ & + \boldsymbol{\varphi}_x'\boldsymbol{m}_x u_t \end{aligned} \tag{E.13}$$

This will fit the proposed solution if

$$a_{31} = \rho \boldsymbol{\varphi}_x' \boldsymbol{m}_x \tag{E.14a}$$

$$a_{32} = \theta \boldsymbol{\varphi}_x' \boldsymbol{m}_p \tag{E.14b}$$

$$a_{33} = a_{33} + \boldsymbol{\varphi}_x' \boldsymbol{m}_p \left(\frac{1-\varphi}{\varphi} \right) \gamma \tag{E.14c}$$

$$a_{34} = a_{34} + \boldsymbol{\varphi}_x' \boldsymbol{m}_p \left(\frac{1-\varphi}{\varphi} \right) (\lambda - \theta) \tag{E.14d}$$

$$0 = (1-\varphi)\rho + a_{31}\varphi - a_{33}(1-\varphi) - \boldsymbol{\varphi}_x' \boldsymbol{m}_x \rho + \boldsymbol{\varphi}_x' \boldsymbol{m}_p \frac{\gamma}{\varphi}(1-\varphi) \tag{E.14e}$$

$$0 = a_{32}\varphi - a_{34}(1-\varphi) - \boldsymbol{\varphi}_x' \boldsymbol{m}_p \theta + \boldsymbol{\varphi}_x' \boldsymbol{m}_p \frac{\lambda - \theta}{\varphi}(1-\varphi) \tag{E.14f}$$

$$b_3 = \boldsymbol{\varphi}_x' \boldsymbol{m}_x \tag{E.14g}$$

Combining (E.14a), (E.14c) and (E.14e) then gives

$$a_{33} = \rho \left(1 - \boldsymbol{\varphi}_x' \boldsymbol{m}_x\right) \tag{E.15}$$

While combining (E.14b), (E.14d) and (E.14f) gives

$$a_{34} = -\theta \boldsymbol{\varphi}_x' \boldsymbol{m}_p \tag{E.16}$$

E.3.2 The law of motion for $\widetilde{p}_{t-1|t}$

Stepping (E.7) back one period, we can expand (E.10b) to read:

$$\begin{aligned} \widetilde{p}_{t-1|t} = \left(\boldsymbol{\varphi}_p' A - \boldsymbol{\varphi}_p' MNA\right) & \begin{bmatrix} \overline{E}_{t-1}\left[x_{t-1}\right] \\ \overline{E}_{t-1}\left[p_{t-2}\right] \\ \frac{1}{\varphi}\left(\widetilde{x}_{t-1|t-1} - (1-\varphi)\overline{E}_{t-1}\left[x_{t-1}\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{p}_{t-2|t-1} - (1-\varphi)\overline{E}_{t-1}\left[p_{t-2}\right]\right) \end{bmatrix} \\ & + \boldsymbol{\varphi}_p' MNAZ_{t-1} + \boldsymbol{\varphi}_p' MNBu_t \end{aligned} \tag{E.17}$$

Expanding $\boldsymbol{\varphi}_x' A$ and NA and NB, and then gathering like terms, this gives:

$$\widetilde{p}_{t-1|t} = \left\{\boldsymbol{\varphi}_p' \boldsymbol{m}_x \rho\right\} x_{t-1} \tag{E.18}$$

$$\begin{aligned} & + \left\{\boldsymbol{\varphi}_p' \boldsymbol{m}_p \theta\right\} p_{t-2} \\ & + \left\{\left(\frac{\gamma(1-\varphi) + a_{43}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \gamma\right\} \widetilde{x}_{t-1|t-1} \\ & + \left\{\left(\frac{(\lambda-\theta)(1-\varphi) + a_{44}\varphi}{\varphi}\right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\lambda-\theta}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p (\lambda-\theta)\right\} \widetilde{p}_{t-2|t-1} \\ & + \left\{a_{41}\varphi - \left(\frac{\gamma(1-\varphi) + a_{43}\varphi}{\varphi}\right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_x \rho + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\left[x_{t-1}\right] \\ & + \left\{\theta(1-\varphi) + a_{42}\varphi - \left(\frac{(\lambda-\theta)(1-\varphi) + a_{44}\varphi}{\varphi}\right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \theta + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{(\lambda-\theta)}{\varphi}(1-\varphi)\right\} \overline{E}_{t-1}\left[p_{t-2}\right] \end{aligned} \tag{E.19}$$

$$+ \boldsymbol{\varphi}_p' \boldsymbol{m}_x u_t$$

This will fit the proposed solution if

$$a_{41} = \rho \boldsymbol{\varphi}_p' \boldsymbol{m}_x \tag{E.20a}$$

$$a_{42} = \theta \boldsymbol{\varphi}_p' \boldsymbol{m}_p \tag{E.20b}$$

$$a_{43} = \left(\frac{\gamma (1-\varphi) + a_{43}\varphi}{\varphi} \right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \gamma \tag{E.20c}$$

$$a_{44} = \left(\frac{(\lambda-\theta)(1-\varphi) + a_{44}\varphi}{\varphi} \right) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\lambda-\theta}{\varphi} + \boldsymbol{\varphi}_p' \boldsymbol{m}_p (\lambda-\theta) \tag{E.20d}$$

$$0 = a_{41}\varphi - \left(\frac{\gamma (1-\varphi) + a_{43}\varphi}{\varphi} \right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_x \rho + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{\gamma}{\varphi}(1-\varphi) \tag{E.20e}$$

$$0 = \theta(1-\varphi) + a_{42}\varphi - \left(\frac{(\lambda-\theta)(1-\varphi) + a_{44}\varphi}{\varphi} \right)(1-\varphi) - \boldsymbol{\varphi}_p' \boldsymbol{m}_p \theta + \boldsymbol{\varphi}_p' \boldsymbol{m}_p \frac{(\lambda-\theta)}{\varphi}(1-\varphi) \tag{E.20f}$$

$$b_4 = \boldsymbol{\varphi}_p' \boldsymbol{m}_x \tag{E.20g}$$

Combining (E.20a), (E.20c) and (E.20e) then gives

$$a_{43} = \gamma - \rho \boldsymbol{\varphi}_p' \boldsymbol{m}_x \tag{E.21}$$

While combining (E.20b), (E.20d) and (E.20f) gives

$$a_{44} = \lambda - \theta \boldsymbol{\varphi}_p' \boldsymbol{m}_p \tag{E.22}$$

E.3.3 The overall law of motion

For given values of φ, M and $\boldsymbol{\alpha}$, the law of motion is therefore given by:

$$Z_t = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \theta & \gamma & \lambda-\theta \\ \rho \boldsymbol{\varphi}_x' \boldsymbol{m}_x & \theta \boldsymbol{\varphi}_x' \boldsymbol{m}_p & \rho(1-\boldsymbol{\varphi}_x' \boldsymbol{m}_x) & -\theta \boldsymbol{\varphi}_x' \boldsymbol{m}_p \\ \rho \boldsymbol{\varphi}_p' \boldsymbol{m}_x & \theta \boldsymbol{\varphi}_p' \boldsymbol{m}_p & \gamma - \rho \boldsymbol{\varphi}_p' \boldsymbol{m}_x & \lambda - \theta \boldsymbol{\varphi}_p' \boldsymbol{m}_p \end{bmatrix}}_{A} Z_{t-1} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \boldsymbol{\varphi}_x' \boldsymbol{m}_x \\ \boldsymbol{\varphi}_p' \boldsymbol{m}_x \end{bmatrix}}_{B} u_t \tag{E.23}$$

E.4 The equilibrium degree of strategic complementarity (φ)

The next step is to find φ (the weight used in constructing $\widetilde{x}_{t|t}$ and $\widetilde{p}_{t-1|t}$). We have that

$$p_t = \boldsymbol{\alpha}' Z_t = \theta p_{t-1} + (\lambda-\theta)\widetilde{p}_{t-1|t} + \gamma \widetilde{x}_{t|t} \tag{E.24}$$

Given A, firms' average expectation of the next-period price level is therefore given by:

$$\overline{E}_t[p_{t+1}] = \boldsymbol{\alpha}' \overline{E}_t[Z_{t+1}] = \boldsymbol{\alpha}' \overline{E}_t \begin{bmatrix} x_{t+1} \\ p_t \\ \widetilde{x}_{t+1|t+1} \\ \widetilde{p}_{t|t+1} \end{bmatrix} = \boldsymbol{\alpha}' \overline{E}_t \begin{bmatrix} \boldsymbol{a}_1' Z_t \\ p_t \\ \boldsymbol{a}_2' Z_t \\ \boldsymbol{a}_4' Z_t \end{bmatrix} = \boldsymbol{\alpha}' \left(\boldsymbol{e}_2 \overline{E}_t[p_t] + J_2 A \overline{E}_t[Z_t] \right)$$

where $\boldsymbol{e}_2$ is a column vector of zeros with a one in the third position, and J_2 is the identity matrix modified to put a zero in the third position of the lead diagonal. Since $\overline{E}_t\left[Z_{t+q}\right] = A^{q-1}\overline{E}_t\left[Z_{t+1}\right]$, it follows that:

$$\overline{E}_t\left[p_{t+q}\right] = \boldsymbol{\alpha}' A^{q-1}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \tag{E.25}$$

Substituting (E.25) into the competitive equilibrium condition (D.2) then gives

$$\begin{aligned} p_t = & \quad \theta\, p_{t-1} \\ & + \begin{bmatrix} b_p & \zeta_{-1} & 0 & 0 \end{bmatrix} \overline{E}_t\left[Z_t\right] \\ & + \quad \zeta_0\, \overline{E}_t\left[p_t\right] \\ & + \quad \beta\theta\, \boldsymbol{\alpha}'\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \\ & + \quad \zeta_{1^+}\ (1-\delta)\sum_{q=1}^{\infty}\delta^{q-1}\boldsymbol{\alpha}' A^{q-1}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \end{aligned} \tag{E.26}$$

Or, gathering like terms,

$$p_t = \theta p_{t-1} + \boldsymbol{d}'\overline{E}_t\left[Z_t\right] + \varphi\overline{E}_t\left[p_t\right] \tag{E.27a}$$

where

$$\boldsymbol{d}' = \begin{bmatrix} b_p & \zeta_{-1} & 0 & 0 \end{bmatrix} + \boldsymbol{\alpha}'\left(\beta\theta + \zeta_{1^+}(1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right) J_2 A \tag{E.27b}$$

$$\varphi = \zeta_0 + \beta\theta\boldsymbol{\alpha}'\boldsymbol{e}_2 + \zeta_{1^+}\boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}(\delta A)^q\right)\boldsymbol{e}_2 \tag{E.27c}$$

The coefficient φ is the *equilibrium degree of strategic complementarity* in firms' price-setting decisions (that is, after taking account of demand and the entire expected future path of prices). Expanding the compound parameters $\boldsymbol{\alpha}'\boldsymbol{e}_2$, ζ_0 and ζ_{1^+}, equation (E.27c) may then be rewritten as:

$$\varphi = (1-\theta)(1-\beta\theta)\left(1 - \sigma\omega\delta\left(\phi_\pi + (1-\phi_\pi\delta)\left(1 - \boldsymbol{\alpha}'\left((1-\delta)\sum_{q=0}^{\infty}\left(\delta A(\varphi)\right)^q\right)\boldsymbol{e}_2\right)\right)\right) \tag{E.28}$$

where I have emphasised that the transition matrix A is itself a function of φ.

E.5 Bringing everything together

We then have that, conditional on a particular forward-looking solution under full information (λ, γ), the law of motion is a function of the Kalman gain and the strategic complementarity $(A = f(M, \varphi))$; the Kalman gain is a function of the law of motion $(M = g(A))$; and the strategic complementarity is a function of the law of motion $(\varphi = h(A))$. The solution is then the fixed point of equations (E.4), (E.23) and (E.28): $A = f(g(A), h(A))$.

F Uniqueness

F.1 Proof of proposition 6

To begin, recall that the equilibrium conditions of the model are:

$$p_t = \theta p_{t-1} + (1 - \theta(1+\beta))\,\overline{E}_t[p_t] + \beta\theta\overline{E}_t[p_{t+1}] + \kappa\theta\overline{E}_t[y_t] \tag{F.1a}$$

$$y_t = \delta E_t^{\Omega}[y_{t+1}] - \delta\sigma\left(\phi_\pi(p_t - p_{t-1}) - \left(E_t^{\Omega}[p_{t+1}] - p_t\right) - x_t\right) \tag{F.1b}$$

and the purely forward-looking solution is of the form:

$$Z_t = AZ_{t-1} + Bu_t \tag{F.2a}$$

$$\boldsymbol{\zeta}_t = \boldsymbol{\alpha}' Z_t \tag{F.2b}$$

The candidate solution, including the bubble term, is:

$$\boldsymbol{\zeta}_t = \boldsymbol{\alpha}' Z_t + \boldsymbol{\epsilon}_t \tag{F.3}$$

$$A_0\boldsymbol{\epsilon}_t = A_1 E_t^{\Omega}[\boldsymbol{\epsilon}_{t+1}] + B_1\boldsymbol{\epsilon}_{t-1} \tag{F.4}$$

where A_0, A_1 and B_1 are defined in the main text. Define $\mu \in (0,1)$ and suppose that firms' period-t signals about $\boldsymbol{\epsilon}$ are given by:[42]

$$\boldsymbol{s}_t^{\epsilon}(i) = \begin{cases} \boldsymbol{\epsilon}_t + \boldsymbol{v}_t^{\epsilon}(i) & \text{if} \quad i \in [0,\mu) \\ \boldsymbol{\epsilon}_t & \text{if} \quad i \in [\mu,1] \end{cases} \tag{F.5}$$

where $\boldsymbol{v}_t^{\epsilon}(i) \sim i.i.d.\, N(\mathbf{0}, \sigma_\epsilon^2 I_2)$. Substituting (F.3) into (F.1) gives:

$$p_t = \boldsymbol{\alpha}_p' Z_t + \theta\epsilon_{t-1}^p + (1 - \theta(1+\beta))\,\overline{E}_t[\epsilon_t^p] + \beta\theta\overline{E}_t[\epsilon_{t+1}^p] + \kappa\theta\overline{E}_t[\epsilon_t^y] \tag{F.6a}$$

$$y_t = \boldsymbol{\alpha}_y' Z_t + \delta E_t^{\Omega}[\epsilon_{t+1}^y] - \delta\sigma\left(\phi_\pi(\epsilon_t^p - \epsilon_{t-1}^p) - \left(E_t^{\Omega}[\epsilon_{t+1}^p] - \epsilon_t^p\right)\right) \tag{F.6b}$$

By comparison to equation (F.3) and with some rearranging, this implies that:

$$A_0\boldsymbol{\epsilon}_t = B_1\boldsymbol{\epsilon}_{t-1} + A_1 E_t^{\Omega}[\boldsymbol{\epsilon}_{t+1}] + \mu\widetilde{A}_0\left\{\overline{E}_t^{\epsilon}[\boldsymbol{\epsilon}_t] - \boldsymbol{\epsilon}_t\right\} + \mu\widetilde{A}_1\left\{\overline{E}_t^{\epsilon}[\boldsymbol{\epsilon}_{t+1}] - E_t^{\Omega}[\boldsymbol{\epsilon}_{t+1}]\right\} \tag{F.7a}$$

where $\overline{E}_t^{\epsilon}[\cdot]$ is the average expectation of those firms with noisy signals about $\boldsymbol{\epsilon}_t$ and

$$\widetilde{A}_0 = \begin{bmatrix} (1+\theta(1+\beta)) & \theta\kappa \\ 0 & 0 \end{bmatrix} \qquad \widetilde{A}_1 = \begin{bmatrix} \beta\theta & 0 \\ 0 & 0 \end{bmatrix} \tag{F.7b}$$

For the bubble to feature in the solution, the last two terms in (F.7a) must equal zero. But this can only happen if the share of agents with noisy signals (μ) falls to zero, or if the variance of the noise they face (σ_ϵ) falls to zero (so that expectation errors are zero), both of which would imply universal full knowledge of the bubble, a contradiction.

[42] These are in addition to signals about $\boldsymbol{\eta}_t$.

F.2 Proof of proposition 7

Let $\mathcal{S}(\cdot)$ be a set of selection matrices for extracting elements from Y_t. For example, $\mathcal{S}(x_t) = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}$ so that $\mathcal{S}(x_t) Y_t = x_t$. Similarly, $\mathcal{S}\begin{pmatrix} x_{t-1} \\ \zeta_{t-2} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \end{bmatrix}$ so that $\mathcal{S}\begin{pmatrix} x_{t-1} \\ \zeta_{t-2} \end{pmatrix} Y_t = \begin{bmatrix} x_{t-1} \\ \zeta_{t-2} \end{bmatrix}$. Further define the selection matrix T such that $TY_t = \overline{E}_t[Y_t]$. Note that the matrix T amounts to a shift operator. Pre-multiplying a matrix by T will shift its elements up $3(1+H)$ places. Pre-multiplying by T' will shift elements down by the same amount. Post-multiplying by T shifts a matrix right, while post-multiplying by T' shifts a matrix left.

Substituting the candidate solution (53) into the equilibrium conditions (F.1) gives:

$$\boldsymbol{d}_p' Y_t = \theta \mathcal{S}(p_{t-1}) Y_t + (1 - \theta(1+\beta)) \boldsymbol{d}_p' T Y_t + \beta\theta \boldsymbol{d}_p' F T Y_t + \kappa\theta \boldsymbol{d}_y' T Y_t \tag{F.8a}$$

$$\boldsymbol{d}_y' Y_t = \delta \boldsymbol{d}_y' F Y_t - \delta\sigma \left(\phi_\pi \left(\boldsymbol{d}_p' Y_t - \mathcal{S}(p_{t-1}) Y_t \right) - \left(\boldsymbol{d}_p' F Y_t - \boldsymbol{d}_p' Y_t \right) - \mathcal{S}(x_t) Y_t \right) \tag{F.8b}$$

Stacking the two equations, dropping the Y_ts and gathering like terms then gives:

$$\begin{aligned} D = \begin{bmatrix} 0 & \theta \\ \delta\sigma & \delta\sigma\phi_\pi \end{bmatrix} \begin{bmatrix} \mathcal{S}(x_t) \\ \mathcal{S}(p_{t-1}) \end{bmatrix} &+ \begin{bmatrix} 0 & 0 \\ -\delta\sigma(1+\phi_\pi) & 0 \end{bmatrix} D + \begin{bmatrix} (1-\theta(1+\beta)) & \kappa\theta \\ 0 & 0 \end{bmatrix} DT \\ &+ \begin{bmatrix} 0 & 0 \\ \delta\sigma & \delta \end{bmatrix} DF \qquad + \begin{bmatrix} \beta\theta & 0 \\ 0 & 0 \end{bmatrix} DFT \end{aligned} \tag{F.9}$$

With a linear model, firms make use of a Kalman filter to form their expectations of Y_t:

$$\begin{aligned} \overline{E}_t[Y_t] &= \overline{E}_{t-1}[Y_t] + \underbrace{\begin{bmatrix} \boldsymbol{k}_x & \boldsymbol{k}_p \end{bmatrix}}_{K} \left\{ \boldsymbol{s}_t(i) - \overline{E}_{t-1}[\boldsymbol{s}_t(i)] \right\} \\ &= FTY_{t-1} + K\mathcal{S}\begin{pmatrix} x_t \\ p_{t-1} \end{pmatrix} F(I-T) Y_{t-1} \end{aligned} \tag{F.10}$$

where $K = \begin{bmatrix} \boldsymbol{k}_x & \boldsymbol{k}_p \end{bmatrix}$ is a matrix of Kalman gains applied against firms' (noisy) signals regarding x_t and p_{t-1} respectively. The state vector Y_t therefore follows an AR(1) process:

$$Y_t = FY_{t-1} + Gu_t \tag{F.11}$$

where the transition matrix satisfies:

$$F = \left[\begin{array}{ccccc|ccccc|c} \rho & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ D_{x,0}^{(0)} & D_{p,0}^{(0)} & \cdots & D_{x,H}^{(0)} & D_{p,H}^{(0)} & D_{x,0}^{(1)} & D_{p,0}^{(1)} & \cdots & D_{x,H}^{(1)} & D_{p,H}^{(1)} & \cdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & I & & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ & & \ddots & & & & & & & & \\ \hline & & & & & FT + K\mathcal{S}\begin{pmatrix} x_t \\ p_{t-1} \end{pmatrix} F(I-T) & & & & & & \end{array} \right] \tag{F.12}$$

Making use of the various selection matrices, this can be rewritten as:

$$F = Q + T'FT = \sum_{k=0}^{\infty} (T')^k Q (T)^k \tag{F.13a}$$

$$Q = \mathcal{S}(x_t)' \rho \mathcal{S}(x_t) + T'\boldsymbol{k}_x \rho \mathcal{S}(x_t)(I-T) + \mathcal{S}\begin{pmatrix} x_{t-1} \\ \zeta_{t-2} \end{pmatrix}' \mathcal{S}\begin{pmatrix} x_t \\ \zeta_{t-1} \end{pmatrix} + \mathcal{S}(p_{t-1})' \boldsymbol{d}_p' + T'\boldsymbol{k}_p \boldsymbol{d}_p'(I-T) \tag{F.13b}$$

For presentational simplicity, I assume for the remainder of this proof that $H = 1$ (that is, considering only solutions as functions of $\begin{bmatrix} x_t & \zeta_{t-1}' & x_{t-1} & \zeta_{t-2}' \end{bmatrix}'$ and higher-order expectations of the same). This nests the complete set of all backward-looking solutions under full information (47) when firms' idiosyncratic noise is taken to zero and, in any event, the logic of the proof is identical for all $H \geq 1$. In this case, I can rewrite D as:

$$D = \left[\begin{array}{cccccc|cccccc|c} d_{p,1} & d_{p,2} & d_{p,3} & d_{p,4} & d_{p,5} & d_{p,6} & d_{p,7} & d_{p,8} & d_{p,9} & d_{p,10} & d_{p,11} & d_{p,12} & \cdots \\ d_{y,1} & d_{y,2} & d_{y,3} & d_{y,4} & d_{y,5} & d_{y,6} & d_{y,7} & d_{y,8} & d_{y,9} & d_{y,10} & d_{y,11} & d_{y,12} & \cdots \end{array}\right] \tag{F.14}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Sub-block (0)}}\underbrace{\qquad\qquad\qquad\qquad}_{\text{Sub-block (1)}}$$

where a rejection of backward-looking solutions must demonstrate that columns 4, 5, 6, 10, 11, 12, etc are all zeros. First, consider $\begin{bmatrix} d_{p,4} & d_{p,5} & d_{p,6} \end{bmatrix}$. Since (i) $\mathcal{S}(x_t)$ and $\mathcal{S}(p_{t-1})$ have zeros everywhere from column 3 onwards; (ii) the top row of coefficients in the second and third terms of (F.9) are zero; and (iii) the fourth and fifth terms of (F.9) are both post-multiplied by T (shifting them to the right), we must have:

$$D = \left[\begin{array}{cccccc|cccccc|c} d_{p,1} & d_{p,2} & d_{p,3} & 0 & 0 & 0 & d_{p,7} & d_{p,8} & d_{p,9} & d_{p,10} & d_{p,11} & d_{p,12} & \cdots \\ d_{y,1} & d_{y,2} & d_{y,3} & d_{y,4} & d_{y,5} & d_{y,6} & d_{y,7} & d_{y,8} & d_{y,9} & d_{y,10} & d_{y,11} & d_{y,12} & \cdots \end{array}\right] \tag{F.15}$$

Second, consider $\begin{bmatrix} d_{y,4} & d_{y,5} & d_{y,6} \end{bmatrix}$. If we define $R \equiv DQ$, which is the same dimension as D, examination of (F.9) implies that $\begin{bmatrix} d_{y,4} & d_{y,5} & d_{y,6} \end{bmatrix} = \delta \begin{bmatrix} r_{y,4} & r_{y,5} & r_{y,6} \end{bmatrix}$. But combining (F.15) and (F.13), I obtain $\begin{bmatrix} r_{y,4} & r_{y,5} & r_{y,6} \end{bmatrix} = \boldsymbol{d}_y'(\boldsymbol{s}_p + T'\boldsymbol{k}_p)\begin{bmatrix} d_{p,4} & d_{p,5} & d_{p,6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$:[43]

$$D = \left[\begin{array}{cccccc|cccccc|c} d_{p,1} & d_{p,2} & d_{p,3} & 0 & 0 & 0 & d_{p,7} & d_{p,8} & d_{p,9} & d_{p,10} & d_{p,11} & d_{p,12} & \cdots \\ d_{y,1} & d_{y,2} & d_{y,3} & 0 & 0 & 0 & d_{y,7} & d_{y,8} & d_{y,9} & d_{y,10} & d_{y,11} & d_{y,12} & \cdots \end{array}\right] \tag{F.16}$$

Third, consider $\begin{bmatrix} d_{p,10} & d_{p,11} & d_{p,12} \end{bmatrix}$. Combining (F.16), (F.13) and (F.9), it follows that these must equal $\begin{bmatrix} (1-\theta(1+\beta)) & \kappa\theta \end{bmatrix} \begin{bmatrix} d_{p,4} & d_{p,5} & d_{p,6} \\ d_{y,4} & d_{y,5} & d_{y,6} \end{bmatrix} + \beta\theta \begin{bmatrix} r_{p,4} & r_{p,5} & r_{p,6} \end{bmatrix}$, which given the earlier results, must equal zero.

$$D = \left[\begin{array}{cccccc|cccccc|c} d_{p,1} & d_{p,2} & d_{p,3} & 0 & 0 & 0 & d_{p,7} & d_{p,8} & d_{p,9} & 0 & 0 & 0 & \cdots \\ d_{y,1} & d_{y,2} & d_{y,3} & 0 & 0 & 0 & d_{y,7} & d_{y,8} & d_{y,9} & d_{y,10} & d_{y,11} & d_{y,12} & \cdots \end{array}\right] \tag{F.17}$$

The same logic then continues to $\begin{bmatrix} d_{y,10} & d_{y,11} & d_{y,12} \end{bmatrix}$, followed by $\begin{bmatrix} d_{p,16} & d_{p,17} & d_{p,18} \end{bmatrix}$ and so forth. Each sub-block, thanks to the shifting by T^k, depends on the values of $\begin{bmatrix} d_{p,4} & d_{p,5} & d_{p,6} \end{bmatrix}$, which was established to be zeros above.

[43] Similarly, $\begin{bmatrix} r_{p,4} & r_{p,5} & r_{p,6} \end{bmatrix} = \boldsymbol{d}_p'(\boldsymbol{s}_p + T'\boldsymbol{k}_p)\begin{bmatrix} d_{p,4} & d_{p,5} & d_{p,6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

G Non-zero trend inflation

Although appendix C derived the Phillips curve under ICK with non-zero trend inflation, subsequent analysis was then done with an assumption of zero trend inflation. This appendix presents the full solution with positive trend inflation.

G.1 The purely forward-looking solution under full info

Granting full information to price-setting firms in (C.27) - (C.32), the system can be stacked and rearranged to:

$$\mathcal{A}\begin{bmatrix} p_{t-1} \\ y_{t-1} \\ \psi_{t-1} \\ \phi_{t-1} \\ s_{t-1} \end{bmatrix} + \mathcal{B}\begin{bmatrix} p_{t} \\ y_{t} \\ \psi_{t} \\ \phi_{t} \\ s_{t} \end{bmatrix} + \mathcal{C}E_t^{\Omega}\begin{bmatrix} p_{t+1} \\ y_{t+1} \\ \psi_{t+1} \\ \phi_{t+1} \\ s_{t+1} \end{bmatrix} + \mathcal{D}x_t = 0 \tag{G.1}$$

where

$$\mathcal{B} = \begin{bmatrix} -1 & 0 & \left(\frac{\left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)(1-\alpha)}{1-\alpha+\alpha\varepsilon}\right) & -\left(\frac{\left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)(1-\alpha)}{1-\alpha+\alpha\varepsilon}\right) & 0 \\ -\sigma\left(1+\phi_\pi\right) & -\left(1+\sigma\phi_y\right) & 0 & 0 & 0 \\ \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(\frac{\varepsilon}{1-\alpha}\right) & \left(\sigma+\left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(\frac{1}{1-\alpha}\right)\left(1+\frac{1}{\psi}\right)\right) & -1 & 0 & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\frac{1}{\psi} \\ \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}\right)(\varepsilon-1) & \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)\right) & 0 & -1 & 0 \\ \left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}-\overline{\Pi}^{\varepsilon-1}\right) & 0 & 0 & 0 & -1 \end{bmatrix} \tag{G.2}$$

$$\mathcal{A} = \begin{bmatrix} \theta\overline{\Pi}^{\varepsilon-1} & 0 & 0 & 0 & 0 \\ \sigma\phi_\pi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}-\overline{\Pi}^{\varepsilon-1}\right) & 0 & 0 & 0 & \theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} \end{bmatrix} \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 & 0 \\ 0 & -\sigma\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} & \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} & 0 & 0 \\ 0 & -\sigma\beta\theta\overline{\Pi}^{\varepsilon-1} & 0 & \beta\theta\overline{\Pi}^{\varepsilon-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{D} = \begin{bmatrix} 0 \\ -\sigma \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{G.3}$$

The purely forward-looking solution may then be obtained via any of the usual techniques, such as the linear time iteration of Rendahl (2017). Note that finding the purely forward solution does not automatically imply that it is determinate under full information.

G.2 The purely forward-looking solution under ICK

With non-zero trend inflation, the purely forward-looking solution under full information is a function of three variables: x_t, p_{t-1} and s_{t-1}. Under ICK, the full state is:

$$Z_t \equiv \begin{bmatrix} \boldsymbol{\eta}_t \\ \widetilde{\boldsymbol{\eta}}_{t|t} \end{bmatrix} = AZ_{t-1} + Bu_t \tag{G.4}$$

where $\boldsymbol{\eta}_t \equiv \begin{bmatrix} x_t & p_{t-1} & s_{t-1} \end{bmatrix}'$ and $\widetilde{\boldsymbol{\eta}}_{t|t} \equiv (1-\varphi)\sum_{k=1}^{\infty}\varphi^{k-1}\boldsymbol{\eta}_{t|t}^{(k)}$ for some $\varphi \in (-1,1)$ to be determined.

G.2.1 The reduced-form coefficients

Let the purely forward-looking solution under full information be given by:

$$p_t = \gamma\, x_t + \lambda\, p_{t-1} + \delta\, s_{t-1} \tag{G.5}$$

As with the model with zero trend inflation, since $\widetilde{\boldsymbol{\eta}}_{t|t} = \boldsymbol{\eta}_t$ under full information and $\widetilde{\boldsymbol{\eta}}_{t|t}$ approaches $\boldsymbol{\eta}_t$ smoothly as $\sigma_v \to 0$, simple inspection dictates that the purely forward-looking solution for the price level under ICK is:

$$p_t = \theta\overline{\Pi}^{\varepsilon-1}p_{t-1} + \gamma\,\widetilde{x}_{t|t} + \left(\lambda - \theta\overline{\Pi}^{\varepsilon-1}\right)\widetilde{p}_{t-1|t} + \delta\,\widetilde{s}_{t-1|t} \tag{G.6}$$

Combining this with equation (C.32), the solution for the level of price dispersion is then:

$$\begin{aligned} s_t = & \quad -\left(\frac{\varepsilon\theta}{1-\alpha}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)p_{t-1} \\ & \quad + \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)s_{t-1} \\ & + \left(\frac{\gamma}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\frac{\varepsilon\theta}{1-\alpha}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\widetilde{x}_{t|t} \\ & + \left(\frac{\lambda-\theta\overline{\Pi}^{\varepsilon-1}}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\frac{\varepsilon\theta}{1-\alpha}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\widetilde{p}_{t-1|t} \\ & + \left(\frac{\delta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\frac{\varepsilon\theta}{1-\alpha}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \overline{\Pi}^{\varepsilon-1}\right)\widetilde{s}_{t-1|t} \end{aligned} \tag{G.7}$$

G.2.2 The law of motion

The law of motion for Z_t is given by:

$$\begin{bmatrix} x_t \\ p_{t-1} \\ s_{t-1} \\ \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ p_{t-2} \\ s_{t-2} \\ \widetilde{x}_{t-1|t-1} \\ \widetilde{p}_{t-2|t-1} \\ \widetilde{s}_{t-2|t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} u_t \tag{G.8}$$

where I have filled in the top row from the law of motion for x_t. The second and third rows are given by equations (G.6) and (G.7) respectively. Next, from their definitions:

$$\begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} = \underbrace{\begin{bmatrix} (1-\varphi) & 0 & 0 & \varphi & 0 & 0 \\ 0 & (1-\varphi) & 0 & 0 & \varphi & 0 \\ 0 & 0 & (1-\varphi) & 0 & 0 & \varphi \end{bmatrix}}_{\Phi} \overline{E}_t\left[Z_t\right] \tag{G.9}$$

or, rearranging,

$$\overline{E}_t \begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} = \begin{bmatrix} \frac{1}{\varphi}\left(\widetilde{x}_{t|t} - (1-\varphi)\,\overline{E}_t\left[x_t\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{p}_{t-1|t} - (1-\varphi)\,\overline{E}_t\left[p_{t-1}\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{s}_{t-1|t} - (1-\varphi)\,\overline{E}_t\left[s_{t-1}\right]\right) \end{bmatrix} \tag{G.10}$$

Firms' average Kalman filter for Z_t is:

$$\overline{E}_t\left[Z_t\right] = A\overline{E}_{t-1}\left[Z_{t-1}\right] + M\left\{N\left(AZ_{t-1} + Bu_t\right) - NA\overline{E}_{t-1}\left[Z_{t-1}\right]\right\} \tag{G.11}$$

where $M = \begin{bmatrix} \boldsymbol{m}_x & \boldsymbol{m}_p & \boldsymbol{m}_s \end{bmatrix}$ is a (6×3) Kalman gain matrix to be determined and N selects $\boldsymbol{\eta}_t$ from Z_t so that firm i's signal is $\boldsymbol{s}_t(i) = NZ_t + \boldsymbol{v}_t(i)$. Gathering like terms then gives:

$$\begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} = \Phi\left\{\left(I - MN\right)A\overline{E}_{t-1}\left[Z_{t-1}\right] + MNAZ_{t-1} + MNBu_t\right\} \tag{G.12}$$

Using (G.10) to expand $\overline{E}_{t-1}\left[Z_{t-1}\right]$ gives:

$$\begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} = \Phi\left\{\left(A - MNA\right)\begin{bmatrix} \overline{E}_{t-1}\left[x_{t-1}\right] \\ \overline{E}_{t-1}\left[p_{t-2}\right] \\ \overline{E}_{t-1}\left[s_{t-2}\right] \\ \frac{1}{\varphi}\left(\widetilde{x}_{t-1|t-1} - (1-\varphi)\,\overline{E}_{t-1}\left[x_{t-1}\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{p}_{t-2|t-1} - (1-\varphi)\,\overline{E}_{t-1}\left[p_{t-2}\right]\right) \\ \frac{1}{\varphi}\left(\widetilde{s}_{t-2|t-1} - (1-\varphi)\,\overline{E}_{t-1}\left[s_{t-2}\right]\right) \end{bmatrix} + MNAZ_{t-1} + MNBu_t\right\} \tag{G.13}$$

Note that NA and NB are given by:

$$NA = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & \gamma & (\lambda - \theta) & \delta \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} \qquad NB = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{G.14}$$

with the third row already known from equation (G.7), but too complicated to fill in here (I have filled in the zero elements). Equation (G.13) therefore expands as:

$$
\begin{aligned}
\begin{bmatrix} \widetilde{x}_{t|t} \\ \widetilde{p}_{t-1|t} \\ \widetilde{s}_{t-1|t} \end{bmatrix} &= \Phi \left(\begin{bmatrix} \rho \\ -\gamma \left(\frac{1-\varphi}{\varphi}\right) \\ -a_{34} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{41} - a_{44} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{51} - a_{54} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{61} - a_{64} \left(\frac{1-\varphi}{\varphi}\right) \end{bmatrix} \begin{matrix} -\boldsymbol{m}_x \rho \\ + \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p \gamma + \boldsymbol{m}_s a_{34}) \end{matrix} \right) \overline{E}_{t-1} \left[x_{t-1} \right] \\
&+ \Phi \left(\begin{bmatrix} 0 \\ \theta - (\lambda - \theta) \left(\frac{1-\varphi}{\varphi}\right) \\ a_{32} - a_{35} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{42} - a_{45} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{52} - a_{55} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{62} - a_{65} \left(\frac{1-\varphi}{\varphi}\right) \end{bmatrix} \begin{matrix} -(\boldsymbol{m}_p \theta + \boldsymbol{m}_s a_{32}) \\ + \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p (\lambda - \theta) + \boldsymbol{m}_s a_{35}) \end{matrix} \right) \overline{E}_{t-1} \left[p_{t-2} \right] \\
&+ \Phi \left(\begin{bmatrix} 0 \\ -\delta \left(\frac{1-\varphi}{\varphi}\right) \\ a_{33} - a_{36} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{43} - a_{46} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{53} - a_{56} \left(\frac{1-\varphi}{\varphi}\right) \\ a_{63} - a_{66} \left(\frac{1-\varphi}{\varphi}\right) \end{bmatrix} \begin{matrix} -\boldsymbol{m}_s a_{33} \\ + \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p \delta + \boldsymbol{m}_s a_{36}) \end{matrix} \right) \overline{E}_{t-1} \left[s_{t-2} \right] \\
&+ \Phi \left(\begin{matrix} \{\boldsymbol{m}_x \rho\} & x_{t-1} \\ + \{\boldsymbol{m}_p \theta + \boldsymbol{m}_s a_{32}\} & p_{t-2} \\ + \{\boldsymbol{m}_s a_{33}\} & s_{t-2} \\ + \left\{ \frac{1}{\varphi} \boldsymbol{a}_{*4} - \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p \gamma + \boldsymbol{m}_s a_{34}) \right\} & \widetilde{x}_{t-1|t-1} \\ + \left\{ \frac{1}{\varphi} \boldsymbol{a}_{*5} - \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p (\lambda - \theta) + \boldsymbol{m}_s a_{35}) \right\} & \widetilde{p}_{t-2|t-1} \\ + \left\{ \frac{1}{\varphi} \boldsymbol{a}_{*6} - \left(\frac{1-\varphi}{\varphi}\right) (\boldsymbol{m}_p \delta + \boldsymbol{m}_s a_{36}) \right\} & \widetilde{s}_{t-2|t-1} \\ + \{\boldsymbol{m}_x\} & u_t \end{matrix} \right)
\end{aligned}
\tag{G.15}
$$

where $\boldsymbol{a}_{*4}$, $\boldsymbol{a}_{*5}$ and $\boldsymbol{a}_{*6}$ are the 4th, 5th and 6th columns of A respectively. The following elements of A and B can therefore be read off immediately:

$$
\begin{bmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{bmatrix} = \Phi \begin{bmatrix} \boldsymbol{m}_x \rho & (\boldsymbol{m}_p \theta + \boldsymbol{m}_s a_{32}) & \boldsymbol{m}_s a_{33} \end{bmatrix} \tag{G.16}
$$

$$
\begin{bmatrix} b_4 \\ b_5 \\ b_6 \end{bmatrix} = \Phi \, \boldsymbol{m}_x \tag{G.17}
$$

while the terms against $\widetilde{x}_{t-1|t-1}$, $\widetilde{p}_{t-2|t-1}$ and $\widetilde{s}_{t-2|t-1}$ on the last line of (G.15) imply that:

$$\begin{bmatrix} 0 & 0 & 0 \\ \gamma & (\lambda-\theta) & \delta \\ a_{34} & a_{35} & a_{36} \end{bmatrix} = \Phi\left[(\boldsymbol{m}_p\gamma + \boldsymbol{m}_s a_{34}) \quad (\boldsymbol{m}_p(\lambda-\theta) + \boldsymbol{m}_s a_{35}) \quad (\boldsymbol{m}_p\delta + \boldsymbol{m}_s a_{36})\right] \tag{G.18}$$

Setting the terms against $\overline{E}_{t-1}[x_{t-1}]$, $\overline{E}_{t-1}[p_{t-2}]$ and $\overline{E}_{t-1}[s_{t-2}]$ to zero, and combined with the above, then arrives at:

$$\begin{bmatrix} a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \\ a_{64} & a_{65} & a_{66} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ \gamma & \lambda & \delta \\ a_{34} & (a_{32}+a_{35}) & (a_{33}+a_{36}) \end{bmatrix} - \begin{bmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{bmatrix} \tag{G.19}$$

G.2.3 The equilibrium degree of strategic complementarity

Since lagged values of p_t feature in Z_t, expectations of p_{t+j} are given by:

$$\overline{E}_t[p_{t+j}] = \boldsymbol{a}_{2*}A^{j-1}\left(\boldsymbol{e}_2\overline{E}_t[p_t] + J_2A\overline{E}_t[Z_t]\right) \tag{G.20}$$

where $\boldsymbol{a}_{i*}$ is the ith row of A, $\boldsymbol{e}_i$ is a column vector of zeros with a one in the ith row and J_i is an identity matrix with the ith element of the lead diagonals set to zero.

Substitute the Taylor-type rule into the Euler equation:

$$(1+\sigma\phi_y)\,y_t = E_t^{\Omega}[y_{t+1}] + \sigma\left(\phi_\pi p_{t-1} - (1+\phi_\pi)\,p_t + E_t^{\Omega}[p_{t+1}] + x_t\right) \tag{G.21}$$

Stacking the variables that must be forecast *other than* the price level, I have:

$$\begin{bmatrix} (1+\sigma\phi_y) & 0 & 0 \\ \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)\right) & 1 & 0 \\ \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)\right) & 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \psi_t \\ \phi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\sigma\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} & \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} & 0 \\ -\sigma\beta\theta\overline{\Pi}^{\varepsilon-1} & 0 & \beta\theta\overline{\Pi}^{\varepsilon-1} \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \psi_{t+1} \\ \phi_{t+1} \end{bmatrix}$$
$$+ \begin{bmatrix} -\sigma & 0 \\ 0 & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\frac{1}{\psi} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ s_t \end{bmatrix}$$
$$+ \begin{bmatrix} \sigma\phi_\pi & -\sigma(1+\phi_\pi) & \sigma \\ 0 & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(\frac{\varepsilon}{1-\alpha}\right) & 0 \\ 0 & \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}\right)(\varepsilon-1) & 0 \end{bmatrix} \begin{bmatrix} p_{t-1} \\ p_t \\ p_{t+1} \end{bmatrix}$$

or, rearranging slightly:

$$\begin{bmatrix} y_t \\ \psi_t \\ \phi_t \end{bmatrix} = F\begin{bmatrix} y_{t+1} \\ \psi_{t+1} \\ \phi_{t+1} \end{bmatrix} + G\begin{bmatrix} x_t \\ s_t \end{bmatrix} + H\begin{bmatrix} p_{t-1} \\ p_t \\ p_{t+1} \end{bmatrix} \tag{G.22}$$

where

$$F = \begin{bmatrix} \left(\frac{1}{1+\sigma\phi_y}\right) & 0 & 0 \\ -\sigma\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} - \left(\frac{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)}{1+\sigma\phi_y}\right) & \beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} & 0 \\ -\sigma\beta\theta\overline{\Pi}^{\varepsilon-1} - \left(\frac{1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)}{1+\sigma\phi_y}\right) & 0 & \beta\theta\overline{\Pi}^{\varepsilon-1} \end{bmatrix} \tag{G.23}$$

$$G = \begin{bmatrix} -\left(\frac{\sigma}{1+\sigma\phi_y}\right) & 0 \\ \sigma\left(\frac{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)}{1+\sigma\phi_y}\right) & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\frac{1}{\psi} \\ \sigma\left(\frac{1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)}{1+\sigma\phi_y}\right) & 0 \end{bmatrix} \tag{G.24}$$

$$H = \begin{bmatrix} \left(\frac{\sigma\phi_\pi}{1+\sigma\phi_y}\right) & \left(\frac{-\sigma(1+\phi_\pi)}{1+\sigma\phi_y}\right) & \left(\frac{\sigma}{1+\sigma\phi_y}\right) \\ -\sigma\phi_\pi\left(\frac{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)}{1+\sigma\phi_y}\right) & \left(1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)\left(\frac{\varepsilon}{1-\alpha}\right)+\sigma\left(1+\phi_\pi\right)\left(\frac{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)}{1+\sigma\phi_y}\right) & -\sigma\left(\frac{1-\beta\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}(\omega-\sigma)}{1+\sigma\phi_y}\right) \\ -\sigma\phi_\pi\left(\frac{1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)}{1+\sigma\phi_y}\right) & \left(1-\beta\theta\overline{\Pi}^{\varepsilon-1}\right)(\varepsilon-1)+\sigma\left(1+\phi_\pi\right)\left(\frac{1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)}{1+\sigma\phi_y}\right) & -\sigma\left(\frac{1-\beta\theta\overline{\Pi}^{\varepsilon-1}(1-\sigma)}{1+\sigma\phi_y}\right) \end{bmatrix} \tag{G.25}$$

and where the backward-looking variables follow:

$$\begin{bmatrix} x_t \\ s_t \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}} \end{bmatrix}\begin{bmatrix} x_{t-1} \\ s_{t-1} \end{bmatrix} + \underbrace{\left(\frac{\varepsilon}{1-\alpha}\right)\left(\frac{\theta}{1-\theta\overline{\Pi}^{\varepsilon-1}}\right)\left(\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}-\overline{\Pi}^{\varepsilon-1}\right)\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}}_{D}\begin{bmatrix} p_{t-1} \\ p_t \\ p_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t \tag{G.26}$$

so that

$$\overline{E}_t\begin{bmatrix} x_{t+j} \\ s_{t+j} \end{bmatrix} = \begin{bmatrix} \rho^j & 0 \\ 0 & \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^{j+1} \end{bmatrix}\overline{E}_t\begin{bmatrix} x_t \\ s_{t-1} \end{bmatrix} + \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D\overline{E}_t\begin{bmatrix} p_{t+j-1} \\ p_{t+j} \\ p_{t+j+1} \end{bmatrix} \tag{G.27}$$

Taking the period-t average expectation of (G.22) and substituting it forward gives:

$$\overline{E}_t\begin{bmatrix} y_t \\ \psi_t \\ \phi_t \end{bmatrix} = \sum_{j=0}^{\infty} F^j\overline{E}_t\left[G\begin{bmatrix} x_{t+j} \\ s_{t+j} \end{bmatrix} + H\begin{bmatrix} p_{t+j-1} \\ p_{t+j} \\ p_{t+j+1} \end{bmatrix}\right] \tag{G.28}$$

Substituting in for the backward-looking variables then gives:

$$\overline{E}_t\begin{bmatrix} y_t \\ \psi_t \\ \phi_t \end{bmatrix} = \sum_{j=0}^{\infty} F^j\left(G\begin{bmatrix} \rho^j & 0 \\ 0 & \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^{j+1} \end{bmatrix}\overline{E}_t\begin{bmatrix} x_t \\ s_{t-1} \end{bmatrix} + \left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D + H\right)\overline{E}_t\begin{bmatrix} p_{t+j-1} \\ p_{t+j} \\ p_{t+j+1} \end{bmatrix}\right)$$

Noting that x_t and s_{t-1} are part of Z_t and using equation (G.20) to substitute for the

forward price levels, I obtain:

$$\overline{E}_t \begin{bmatrix} y_t \\ \psi_t \\ \phi_t \end{bmatrix} = \sum_{j=0}^{\infty} F^j G \begin{bmatrix} \rho^j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^{j+1} & 0 & 0 & 0 \end{bmatrix} \overline{E}_t\left[Z_t\right]$$

$$+ \left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D + H\right) \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \overline{E}_t\left[Z_t\right] \\ \overline{E}_t\left[p_t\right] \\ \boldsymbol{a}_{2*}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \end{bmatrix}$$

$$+ F\left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D + H\right) \begin{bmatrix} \overline{E}_t\left[p_t\right] \\ \boldsymbol{a}_{2*}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \\ \boldsymbol{a}_{2*}A\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \end{bmatrix}$$

$$+ \sum_{j=2}^{\infty} F^j \left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D + H\right) \begin{bmatrix} \boldsymbol{a}_{2*}A^{j-2}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \\ \boldsymbol{a}_{2*}A^{j-1}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \\ \boldsymbol{a}_{2*}A^{j}\left(\boldsymbol{e}_2\overline{E}_t\left[p_t\right] + J_2 A\overline{E}_t\left[Z_t\right]\right) \end{bmatrix}$$

Finally, gathering terms and combining this with equation (C.29) obtains:

$$p_t = \theta\overline{\Pi}^{\varepsilon-1} p_{t-1} + \boldsymbol{d}'\,\overline{E}_t\left[Z_t\right] + \varphi\overline{E}_t\left[p_t\right] \tag{G.29}$$

where

$$\varphi = \left(\frac{\left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)(1-\alpha)}{1-\alpha+\alpha\varepsilon}\right) \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \left\{ \begin{array}{l} (GD+H)\begin{bmatrix} 0 \\ 1 \\ \theta \end{bmatrix} \\ +F\left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)D+H\right)\begin{bmatrix} 1 \\ \theta \\ \boldsymbol{a}_{2*}A\boldsymbol{e}_2 \end{bmatrix} \\ +\sum_{j=2}^{\infty} F^j\left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D+H\right)\begin{bmatrix} \boldsymbol{a}_{2*}A^{j-2} \\ \boldsymbol{a}_{2*}A^{j-1} \\ \boldsymbol{a}_{2*}A^{j} \end{bmatrix}\boldsymbol{e}_2 \end{array} \right\} \tag{G.30}$$

$$\boldsymbol{d}' = \left(\frac{\left(1-\theta\overline{\Pi}^{\varepsilon-1}\right)(1-\alpha)}{1-\alpha+\alpha\varepsilon}\right) \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \left\{ \begin{array}{l} \sum_{j=0}^{\infty} F^j G\begin{bmatrix} \rho^j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^{j+1} & 0 & 0 & 0 \end{bmatrix} \\ +(GD+H)\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & \boldsymbol{a}_{2*}J_2A & & & \end{bmatrix} \\ +F\left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)D+H\right)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & \boldsymbol{a}_{2*}J_2A & & & \\ & & \boldsymbol{a}_{2*}AJ_2A & & & \end{bmatrix} \\ +\sum_{j=2}^{\infty} F^j\left(G\left(\theta\overline{\Pi}^{\frac{\varepsilon}{1-\alpha}}\right)^j D+H\right)\begin{bmatrix} \boldsymbol{a}_{2*}A^{j-2} \\ \boldsymbol{a}_{2*}A^{j-1} \\ \boldsymbol{a}_{2*}A^{j} \end{bmatrix} J_2A \end{array} \right\} \tag{G.31}$$

The coefficient φ is the equilibrium degree of strategic complementarity in firms' price-setting problem, taking into account the full effect of general equilibrium and the entire expected path of the price level into the future.

G.3 Uniqueness

The proof of uniqueness when the model has non-zero trend inflation proceeds identically to that under zero trend inflation.

So long as at least some firms observe any candidate bubble with at least some idiosyncratic noise, then the solution cannot feature a rational bubble.

For backward-looking solutions, once the state is defined as:

$$Y_t \equiv \left[\begin{array}{cccccccc} x_t & \boldsymbol{\zeta}'_{t-1} & x_{t-1} & \boldsymbol{\zeta}'_{t-2} & \dots & x_{t-H} & \boldsymbol{\zeta}'_{t-H-1} & \overline{E}_t \left[Y_t\right]' \end{array} \right]' \tag{G.32}$$

so that a solution will be of the form:

$$Y_t = FY_{t-1} + Gu_t \tag{G.33a}$$

$$\boldsymbol{\zeta}_t = DY_t \tag{G.33b}$$

Stacking the system produces an expression in the following form for D:

$$D = \mathcal{A} \begin{bmatrix} \mathcal{S}\left(x_t\right) \\ \mathcal{S}\left(\boldsymbol{\zeta}_{t-1}\right) \end{bmatrix} + \mathcal{B}_0 D + \mathcal{B}_1 DF + \mathcal{C}_0 DT + \mathcal{C}_1 DFT \tag{G.34}$$

where $\mathcal{S}\left(\boldsymbol{\zeta}_{t-1}\right)$ and T are a selection matrices such that $\mathcal{S}\left(\boldsymbol{\zeta}_{t-1}\right) Y_t = \boldsymbol{\zeta}_{t-1}$ from Y_t and $TY_t = \overline{E}_t \left[Y_t\right]$; and $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{C}_0$ and $\mathcal{C}_1$ are matrices of structural parameters.

With firms subject to ICK, the rows of $\mathcal{B}_0$ and $\mathcal{B}_1$ relating to the price level will be all zeros. Since $\mathcal{S}\left(x_t\right)$ and $\mathcal{S}\left(x_t\right)$ have only zeros in columns after $\boldsymbol{\zeta}_{t-1}$ and post-multiplying a matrix by T shifts its elements one 'block' to the right, the coefficients of D for the price level against $\begin{bmatrix} x_{t-1} & \boldsymbol{\zeta}'_{t-2} & \dots & x_{t-H} \end{bmatrix}$ will all be zero. This then implies that coefficients for other endogenous variables against them will also be zero, and subequently that the same applies for expectations of the same.